Adjunctions on the lattice of dendrograms and hierarchies

Fernand Meyer

Adjunctions on the lattice of dendrograms and hierarchies

Fernand Meyer

July 2, 2011

Abstract

Morphological image processing uses two types of trees. The min-tree represents the relations between the regional minima and the various lakes during flooding. As the level of flooding increases in the various lakes, the flooded domain becomes larger.

A second type of tree is used in segmentation and is mainly associated to the watershed transform. The watershed of a topographic surface constitutes a partition of its support. If the relief is flooded, then for increasing levels of floodings, catchment basins merge. The relation of the catchment basins during flooding also obeys a tree structure.

We start by an axiomatic definition of each type of tree, min and max tree being governed by a single axiom ; for nested catchment basins, a second axiom is required.

There is a one to one correspondance between the trees and an ultrametric half distance, as soon one introduces a total order compatible with the inclusion.

Hierarchies obey a complete lattice structure, on which several adjunctions are defined, leading to the construction of morphological filters.

Hierarchies are particular useful for interactive image segmentation, as they constitute a compact representation of all contours of the image, structured in a way that interesting contours are easily extracted.

The last part extends the classical connections and partial connections to the multiscale case and introduces taxonomies.

1 Introduction

Hierarchies are the classical structure for representing a taxinomy. The most famous taxonomy, the Linnaean system classified nature within a nested hierarchy, starting with three kingdoms. Kingdoms were divided into Classes and they, in turn, into Orders, which were divided into Genera (singular: genus), which were divided into Species (singular: species). Below the rank of species he sometimes recognized taxa of a lower (unnamed) rank (for plants these are now called "varieties").

Hierarchies are also useful in the domain of image processing. In the field of mathematical morphology, two basic hierarchies appear. The first is the min-tree structuring the successive lakes of a flooding and its dual counterpart, the max-tree. It has been introduces by Ph.Salembier [8] as a useful condensation of information and support of powerful image filtering methods, based on the pruning of branches of this tree. During flooding, lakes grow and merge ; the flooded area also becomes larger, but does not necessarily occupy the whole domain.

Hierarchies are also at the core of hierarchical segmentation, as they represent in a condensed way nested partitions obtained through image segmentation. Hierarchies appear quite naturally in the field of morphological segmentation, which uses as tool the watershed of gradient images. As a matter of fact, the catchment basins of a topographic surface form a partition. If a basin is flooded and does not contain a regional minimum anymore, it is absorbed by a neighboring basin and vanishes from the segmentation. A hierarchy is hence obtained by considering the catchment basins associated to increasing degrees of flooding, producing for each particular flooding a partition. For increasing floodings, the partitions become coarser and are nested. They structure the image into a multiscale representation ; the nested partitions permit to weight the contours : the importance of a contour being measured by the level of the hierarchy where it disappears [4].

Both types of tree share a common structure, that of a tree. However they differ by their support : successive lakes while flooding a relief do not cover the complete domain of the relief ; furthermore the covered area increases as the lakes grow higher. On the other hand, the catchment basins of this same relief associated to the successive floodings all partition the domain of the relief. In the first case, we speak of dendrogram or partial hierarchy, in the second case of hierarchy or covering hierarchy.

Often one is not interested in partitioning the total domain of the image, but one wants to get the masks of some objects of interest. These masks are disjoint sets but do not partition the domain ; they constitute a partial partition as introduced by Ch. Ronse [5]. The paper is organized as follows.

The first part starts with the axiomatic definition of trees, dendrograms and hierarchies due to Benzecri [2]. Dendrograms are based solely on the intersection axiom and correctly model min and max-trees. Hierarchies are obtained by adding a second axiom, the union axiom.

We show in this paper, that this second axiom very often is not necessary and that most useful properties derive from the intersection axiom alone. Adding the union axiom obliges filling the empty spaces left by an operator like an erosion applied to a hierarchy ; in contrast, the intersection axiom alone allows an automatic adjustment of the support during the erosion.

Dendrograms may be further structured by adding a complete preorder relation, compatible with inclusion order, called stratification level. Stratified hierarchies are the basis of taxonomy. A partial ultrametric distance is then associated to each couple (dendrogram, stratification).

A third part establishes that partial hierarchies form a complete lattice.

The fourth part defines two adjunctions on partial hierarchies. The first extends the adjunction defined by J.Serra for partitions [11], where each tile of a partition is eroded and dilated separately, empty spaces being filled with singletons. Ch. Ronse described the adjoint dilation [6] and also adapted this definition to partial partitions, where the empty spaces are kept outside of the support of the result [7]. We extend the adjunction defined by J.Serra and Ch. Ronse to partial hierarchies. As the support of partial hierarchies and dendrograms may vary, the definition of erosions and dilations is easier on a dendrogram as on a hierarchy.

We also define a second adjunction which directly relies on the complete lattice structure of PUHD. The supremum of translated PUHD yields the erosion by a structuring element equal to the set of translations ; the infimum of translated PUHD yields the dilation. This second adjunction is finer than the first.

In a fifth part, we show how some interactive segmentation tools may be derived from a hierarchy.

The last part of the document extends the connections defined for partitions to hierarchies and defines taxonomies. The algebraic structure of partitions has been studies by Serra, Heijmans, Ronse ([3], [12], [7]). The same set may be partitioned into distinct partitions according the type of connectivity one adopts. Serra has laid down the adequate framework for extending the topological notion of connectivity by defining connective classes later called connections ([10], [12]). Taxonomies, like connections are generated by the union of sets with a non empty intersection ; a taxonomy class possesses an stratification index, which can be interpreted as the diameter for an ultrametric distance. The diameter of a family of sets with an empty intersection being the largest diameter of a set in the family. Connected classes are then simply the sets with diameter 0 for a binary ultrametric distance.

2 Dendrograms

The axiomatic definition of dendrograms and hierarchies is due to Benzecri [2]. It entirely relies on set intersection or union and on the inclusion order relation between sets. The construction is very progressive : starting with the inclusion order relation alone and adding axioms in order to successively define trees, hierarchies and finally stratified hierarchies.

2.1 The structure associated to an order relation

Let E be a domain whose elements are called points. Let $\mathcal{X}$ be a subset of $\mathcal{P}(E)$, on which we consider an arbitrary order or preorder relation relation $\prec$ (the inclusion $\subset$ between sets is an example, but what follows is valid for any preorder relation). The union of all sets belonging to $\mathcal{X}$ is called support of $\mathcal{X}$: $\operatorname{supp}(\mathcal{X})$. The subsets of $\mathcal{X}$ may be structured into:

- the summits : $\operatorname{Sum}(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : A \prec B \Rightarrow A = B\}$

- the leaves : $\text{Leav}(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : B \prec A \Rightarrow A = B\}$
- the nodes : $\text{Nod}(\mathcal{X}) = \mathcal{X} - \text{Leav}(\mathcal{X})$
- the predecessors : $\text{Pred}(A) = \{B \in \mathcal{X} \mid A \prec B\}$
- the immediate predecessors :
  $\text{ImPred}(A) = \{B \in \mathcal{X} \mid \{U \mid U \in \mathcal{X}, A \prec U \text{ and } U \prec B\} = (A, B)\}$
- the successors : $\text{Succ}(A) = \{B \in \mathcal{X} \mid B \prec A\}$
- the immediate successors :
  $\text{ImSucc}(A) = \{B \in \mathcal{X} \mid \{U \mid U \in \mathcal{X}, B \prec U \text{ and } U \prec A\} = (A, B)\}$

The leaves are disjoint sets ; so are also the summits. The summits of $\mathcal{X}$ constitute a partition of $\text{supp}(\mathcal{X})$. This is not necessarily the case of the leaves : a set $B \subset A$ may be a leave, but the remaining points of A do not necessarily belong to a leave. It will only be the case if the union axiom is satisfied, yielding covering hierarchies (see below).

The leaves are successors of the summits and local minima ; the summits are predecessors of the leaves and local maxima. The name predecessor and successor supposes that one explores the filiations between nodes in a direction going from the summits to the leaves, from coarse to fine.

2.2 Dendrograms

We now structure $\mathcal{X}$ as a tree or a dendrogram. We also use "partial hierarchy" as an alternative name for dendrogram.

Dendrograms : $\mathcal{X}$ is a dendrogram if and only if the set $\text{Pred}(A)$ of the predecessors of A, with the order relation induced by $\prec$ is a total order. The maximal element of this family is a summit, which is the unique summit containing A.

There exist several equivalent characterization of dendrograms which are instructive.

Nota bene: From now on we take as preorder relation $\prec$ on $\mathcal{P}(E)$ the ordinary inclusion $\subset$ between sets.

Proposition 1 *The following properties are equivalent:*
1)$\mathcal{X}$ *is a dendrogram*
2) $U, V, A \in \mathcal{X}\ : A \subset U$ *and* $A \subset V \Rightarrow U \subset V$ *or* $V \subset U$
3) $U, V \in \mathcal{X} : U \subsetneq V$ *and* $V \subsetneq U \Rightarrow U \cap V = \varnothing$

Proof.
1) $\Rightarrow$ 2) : Suppose that $\mathcal{X}$ is a dendrogram, i.e. for all $A \in \mathcal{X}$: $\text{Pred}(A)$ is completely ordered for $\subset$
$U, V, A \in \mathcal{X}\ : A \subset U$ and $A \subset V$ means that $U \in \text{Pred}(A)$ and $V \in \text{Pred}(A)$ and since $\text{Pred}(A)$ is completely ordered for $\subset$, we have $U \subset V$ or $V \subset U$
2) $\Rightarrow$ 3) : Suppose now that $U, V, A \in \mathcal{X}\ : A \subset U$ and $A \subset V \Rightarrow U \subset V$ or

$V \subset U$. This implication is equivalent with the following where each predicate has been negated:
$U, V \in \mathcal{X} : U \nsubseteq V$ and $V \nsubseteq U \Rightarrow \nexists A \in \mathcal{X}\ : A \subset U$ and $A \subset V$ But this last predicate implies that $U \cap V$ which is included both in U and in V is empty : $U \cap V = \varnothing$
3) $\Rightarrow$ 2) : Suppose that $U, V \in \mathcal{X} : U \nsubseteq V$ and $V \nsubseteq U \Rightarrow U \cap V = \varnothing$; inverting the predicates yields the equivalent implication $U \cap V \neq \varnothing \Rightarrow U \subset V$ or $V \subset U$. But then if there exists A such that $A \subset U$ and $A \subset V$, it means that $U \cap V \neq \varnothing$ implying $U \subset V$ or $V \subset U$
2) $\Rightarrow$ 1) : Suppose now that $U, V, A \in \mathcal{X}\ : A \subset U$ and $A \subset V \Rightarrow U \subset V$ or $V \subset U$. Consider a set A and two sets $U, V \in \mathrm{Pred}(A)$. This means that $A \subset U$ and $A \subset V$, implying that $U \subset V$ or $V \subset U$, showing that $\mathrm{Pred}(A)$ is indeed well ordered. ■

Proposition 2 *A family $(A_i)_{i \in I}$ of sets in $\mathcal{X}$ with a non empty intersection is completely ordered for $\subset$.*

Proof.
Suppose that there exists a set U included in each A_i. We may then apply the criterion 2 characterizing dendrograms to any couple A_j, A_k showing that $A_j \subset A_k$ or $A_k \subset A_j$ and that the family $(A_i)_{i \in I}$ is completely ordered for $\subset$.
■

Consider now a point p belonging to $\mathrm{supp}(\mathcal{X})$, i.e. there exists a set $A \in \mathcal{X}$ such that $p \in A$. The set $\{p\}$ is included in the family of all sets of $\mathcal{X}$ containing p. This family is thus completely ordered for $\subset$ and contains a smallest element, which we call $\mathrm{home}(p)$. We call $\mathrm{Pred}(p)$ the set of predecessors of $\mathrm{home}(p)$: $\mathrm{Pred}(p) = \mathrm{Pred}(\mathrm{home}(p))$

2.2.1 Representation of a dendrogram as a tree

In the case where $\mathcal{X}$ is finite, then $\mathcal{X}$ is a dendrogram if and only if any element $A \in \mathcal{X} - \mathrm{Sum}(\mathcal{X})$ possesses a unique immediate predecessor.

A dendrogram is said to be connected if it possesses a unique summit : $Card\,\mathrm{Sum}(\mathcal{X}) = 1$

Finite dendrograms are classically represented as a tree : each element $A \in \mathcal{X}$ is a node of the tree, and is linked by an edge with its unique immediate predecessor.

Consider the topographic surface represented in fig.1, flooded by a flood of uniform altitude. As the altitude increases, lakes appear at the position of the regional minima and progressively fill the catchment basins. When the lowest pass point of a catchment basin is reached, two neighboring lakes merge, forming a new lake. The mintree [8] represents the evolution of the lakes during flooding. Each lake is represented as a node : the leaves are the lakes as they appear at the location of the regional minima and are represented as red dots. The lakes created by merging of two preexisting lakes are represented as blue dots. Finally, the unique lake which covers everything is the root of the tree

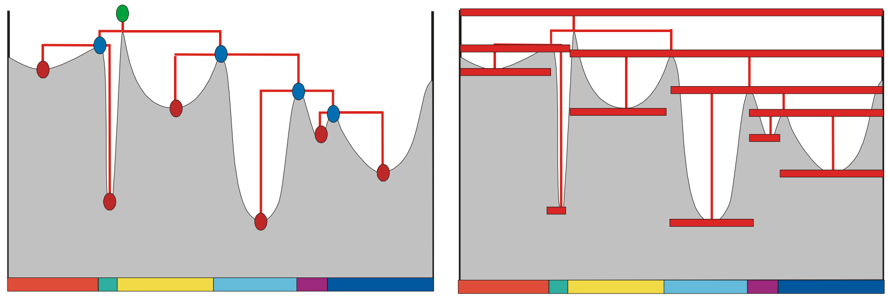

Figure 1: Left : the mintree of a topographic surface. The horizontal mosaic image at the bottom represents the extension of the catchment basins of the relief.
Right : A hierarchy associated to the mintree : each node of the mintree is replaced on the catchment basins of the relief associated to all successors of this node. The altitude of each region represents one possible stratification of this hierarchy

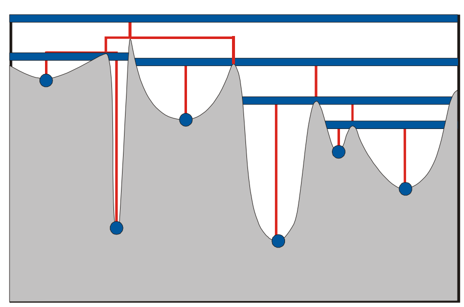

Figure 2: Partial hierarchy of the critical lakes, i.e. the lakes appearing at the minima or the lakes which immediately resulted from the fusion of two smaller lakes.

and is represented as a green dot. Each node, except the summit is linked by an edge with its unique predecessor, the lake formed by merging with another lake. The catchment basins of the relief are represented as coloured mosaic at the bottom of the relief.

We may now associate two distinct families of sets belonging to $\mathcal{P}(E)$. The first one is represented in the right part of figure 1, where each node has been replaced by the union of catchment basins which cut the corresponding minimum or lake. In this case, as one goes down the hierarchy, the regions split but constitute a partition of the domain. The second is represented in fig.2, where each node is replaced by the regional minimum or the lake created at this node. In this second case, as one goes down the hierarchy, the domain covered by the lakes or the minima becomes smaller and smaller. In the binary case, there are only one level : disjoint sets cover the domain E, and constitute a partition. Alternatively, they are disjoint but without covering E, then they constitute a partial partition.

2.2.2 Partitions and partial partitions

Consider a dendrogram Π verifying : $A \in \mathrm{supp}(\Pi) \Rightarrow \mathrm{Pred}(A) = A$. Such a dendrogram is called partial partition (partial partitions have been introduced by Ch. Ronse in [5]). If $\mathrm{supp}(\Pi) = E$, then it is called partition.

Let $U, V \in \diamond$ and $U \neq V$. As $\mathrm{Pred}(U) = U$ and $\mathrm{Pred}(V) = V$ we necessarily have $U \nsubseteqq V$ and $V \nsubseteqq U$, implying according criterion 3 of dendrograms that $U \cap V = \varnothing$.

Inversely consider a subset Π of $\mathcal{P}(E)$ such that any two sets $U, V \in \diamond$ verify $U = V$ or $U \cap V = \varnothing$. Consider now two sets $A, B \in \diamond$ such that $B \in \mathrm{Pred}(A)$. As $A \subset B$, we have $A \cap B \neq \varnothing$, leaving as only possibility $A = B$, showing that $\mathrm{Pred}(A) = A$.

3 Stratification indices and ultrametric half distances

The collection of regions depicted in red in the right part of fig.1 obviously represents a dendrogram: the region at each node is included in all its predecessors. This partial order relation governs the hierarchical structure of the tree. This inclusion order can be made more precise, by the adjunction of a total order compatible with it.

Such a finer partial order between the regions has been introduced in fig.1, where each catchment basin is represented at the altitude of the flooding for which this catchment basin appears for the first time. If a catchment basin is included in another, the its altitude where the first appears is smaller than the altitude where it gets absorbed by the second. For this reason, we say that the altitude constitutes a total order between the catchment basins compatible with the partial inclusion order. We call it a stratification index of the hierarchy. All nodes with the same altitude represent a stratification level of the hierarchy. Let us define precisely what we mean by stratification index.

3.1 Stratification index

$\mathcal{X}$ is a stratified dendrogram (or partial hierarchy), if it is equipped with an index function st from $\mathcal{X}$ into the interval $[0, L]$ of $\mathbb{R}$ which is strictly increasing with the inclusion order:
$\forall A, B \in \mathcal{X} : A \subset B$ and $B \neq A \Rightarrow \mathrm{st}(A) < \mathrm{st}(B)$.
It will be useful to set $\mathrm{st}(\varnothing) = L$. We suppose that for all $A \in \mathcal{X} : \mathrm{st}(A) < L$.

There are many stratification indices compatible with a given hierarchy. Fig.3 presents two stratification indices compatible with the same dendrogram: On the left, we consider the watershed segmentation if one takes as marker the minima, ordered by their altitude. The coarsest level covers the domain. The next level is associated to the two lowest minima taken as markers. The successing levels progressively introduce more minima until all minima are used as markers, cyielding the finest segmentation.
On the right, the coarsest and finest segmentations are the same, in between we consider a flooding with uniform and growing altitude over the domain E. As lakes merge, cachment basins also merge.

This example shows two radically different stratification indices, that is total order, compatible with the partial order induced by inclusionf of sets expressed by the dendrogram.

3.1.1 Extremal stratification indices

Among all possible stratification indices compatible with a hierarchy, there exist two extremal stratification indices.

The largest stratification index assigns the maximal value L to the summits and decreases as one goes down to the successors. The smallest stratification level assigs 0 to alll leaves and increases as one goes up along the predecessors of the leaves.

If the hierarchy $\mathcal{X}$ is finite, the maximal and minimal stratification levels may be computed as follows.

Maximal stratification :

- $\mathrm{st}(summits) = L$
- $\forall A \in \mathcal{X} : \mathrm{st}(A) = \mathrm{st}(\mathrm{ImPred}(A)) - 1$

Minimal stratification :

- $\mathrm{st}(Leaves) = 0$
- $\forall A \in \mathcal{X} : \mathrm{st}(A) = \mathrm{st}(\mathrm{ImSucc}(A)) + 1$

Any linear combination between the maximal and minimal stratification is still a valid stratification.

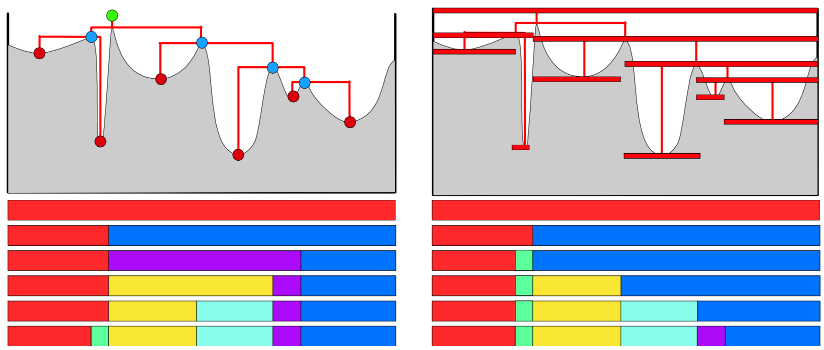

Figure 3: Two stratification indices compatible with the same dendrogram:
On the left, we consider the watershed segmentation if one takes as marker the minima, ordered by their altitude. The coarsest level covers the domain. The next level is associated to the two lowest minima taken as markers. The successing levels progressively introduce more minima until all minima are used as markers, cyielding the finest segmentation.
On the right, the coarsest and finest segmentations are the same. Here we consider a flooding with uniform and growing altitude over the domain E. As lakes merge, cachment basins also merge. The order of merging is however quite different from the left figure.

3.2 A partial ultrametric distance associated to each dendrogram

Definition 3 *χ is a partial ultrametric distance as:*
$\forall p, q \in E : \chi(p, q) = \chi(q, p)$
$\forall p, q, r \in E : \chi(p, q) \leq \max\{\chi(p, r), \chi(r, q)\}$

Proposition 4 *Each dendrogram $\mathcal{X}$ with a stratification index* st *induces on the points of E a partial ultrametric distance χ defined as follows:*

- for $p, q \in E, p \notin \operatorname{supp}(\mathcal{X}) : \chi(p, p) = L$ and $\chi(p, q) = L$
- for $p, q \notin \operatorname{supp}(\mathcal{X})$: if no set of $\mathcal{X}$ contains both p and q, then $\chi(p, q) = L$.
- for $p, q \in \operatorname{supp}(\mathcal{X})$: let A be a set of $\mathcal{X}$ containing both p and q. Thus the family $(A_i)_{i \in I}$ of sets of $\mathcal{X}$ containing both p and q is not empty and has a non empty intersection ; as established above is completely ordered for $\subset$ and possesse a smallest element. The distance $\chi(p, q)$ is the stratification level of the smallest set in this family.

Proof. Let us prove that χ indeed is a partial ultrametric distance.
The symmetry is obvious.
Let us establish the ultrametric inequality $\chi(p, q) \leq \max\{\chi(p, r), \chi(r, q)\}$:

a) if $r \notin \mathrm{supp}(\mathcal{X})$, then $\chi(p,r) = \chi(r,q) = L$ and the ultrametric inequality holds
b) p or q does not belong to $\mathrm{supp}(\mathcal{X})$, say p : then $\chi(p,q) = \chi(p,r) = L$ and the ultrametric inequality holds
c) $p, q, r \in \mathrm{supp}(\mathcal{X})$:
$\chi(p,r)$ is the stratification index of a set $A_1 \in \mathcal{X}$ containing p and r. Hence $A_1 \in \mathrm{Pred}(r)$
Similarly $\chi(r,q)$ is the stratification index of a set $A_2 \in \mathcal{X}$ containing q and r and belongs to $\mathrm{Pred}(r)$.
But $\mathrm{Pred}(r)$ is well ordered for $\subset$, hence $A_1 \subset A_2$ or $A_2 \subset A_1$
Suppose $A_1 \subset A_2$: then A_2 is the smallest set of $\mathrm{Pred}(r)$ containing r and q, but also contains p. Hence $\chi(p,q) \leq \chi(r,q) = \mathrm{st}(A_2)$ since $\chi(p,q)$ is the stratification index of the smallest set of $\mathcal{X}$ containing both p and q. ■

This last inequality is called ultrametric inequality, it is stronger than the triangular inequality.

Definition 5 *For $p \in E$ the closed ball of centre p and radius ρ is defined by* $\mathrm{Ball}(p,\rho) = \{q \in E \mid \chi(p,q) \leq \rho\}$. *The open ball of centre p and radius ρ is defined by* $\overset{\circ}{\mathrm{Ball}}(p,\rho) = \{q \in E \mid \chi(p,q) < \rho\}$.

Remark 6 *Every triangle in a domain where an ultrametric distance is defined is isosceles. Let us consider three distinct points p, q, r and suppose that the largest edge of this triangle is pq. Then $d(p,q) \leq d(p,r) \vee d(r,q)$, showing that the two larges edges of the triangle have the same length.*

3.2.1 Properties of the balls of a partial ultrametric distance

Lemma 7 *Each element of a closed ball* $\mathrm{Ball}(p,\rho)$ *is centre of this ball*

Proof. Suppose that q is an element of $\mathrm{Ball}(p,\rho)$. Let us show that then q also is centre of this ball. If $r \in \mathrm{Ball}(p,\rho) : \chi(q,r) \leq \max\{\chi(q,p), \chi(p,r)\} = \rho$, hence $r \in \mathrm{Ball}(q,\rho)$, showing that $\mathrm{Ball}(p,\rho) \subset \mathrm{Ball}(q,\rho)$. Exchanging the roles of p and q shows that $\mathrm{Ball}(p,\rho) = \mathrm{Ball}(q,\rho)$ ■

Lemma 8 *Two closed balls* $\mathrm{Ball}(p,\rho)$ *and* $\mathrm{Ball}(q,\rho)$ *with the same radius are either disjoint or identical.*

Proof. If $\mathrm{Ball}(p,\rho)$ and $\mathrm{Ball}(q,\rho)$ are not disjoint, then they contain at least one common point r. According to the preceding lemma, r is then centre of both balls $\mathrm{Ball}(p,\rho)$ and $\mathrm{Ball}(q,\rho)$, showing that they are identical. ■

Lemma 9 *The radius of a ball is equal to its diameter.*

Proof. Let $\mathrm{Ball}(p,\rho)$ be a ball of diameter λ, that is the maximal distance between two elements of the ball is λ. Thus $\rho \leq \lambda$. Let q and r be two extremities of a diameter in $\mathrm{Ball}(p,\rho)$: $\lambda = \chi(q,r) \leq \chi(q,p) \vee \chi(p,r) = \rho$. Hence $\lambda = \rho$. ■

Remark 10 *Instead of closed balls, we could have taken open balls. The results are the same.*

3.2.2 Typology of the points of E

Partial hierarchy A stratified dendrogram X structures the domain E into various categories of points :

- a point p is an alien if $p \notin \mathrm{supp}(\mathcal{X})$; for such a point that $\chi(p,p) = L$
- a point p is a singleton if $p \in \mathrm{supp}(\mathcal{X})$ and $\mathrm{home}(p) = \{p\}$; for such a point : $\chi(p,p) < \chi(p,q)$ for $q \neq p$
- all other points of $\mathrm{supp}(\mathcal{X})$ are regular points of $\mathcal{X}$

Due to the ultrametric inequality, we also have $\chi(p,p) \leq \chi(p,q) \vee \chi(q,p) = \chi(p,q)$. Hence $\chi(p,p) \leq \bigwedge\limits_{q \neq p} \chi(p,q)$.

Partial partitions We define aliens and singletons of a partial partition π:

- Singletons are characterized by: $\forall p, q \in E$, $p \neq q$,: $\pi(p,q) = 1$ and $\pi(p,p) = 0$.
- Aliens are characterized by: $\forall p \in E$: $\pi(p,p) = 1$ implying $\forall p, q \in E$: $\pi(p,q) = 1$
- the support of π is the set of points p verifying : $\pi(p,p) = 0$

3.3 Inversely: a dendrogram associated to each partial ultrametric distance

Consider now a partial ultrametric distance χ.

Proposition 11 *The closed balls of a partial ultrametric distance χ form a dendrogram $\mathcal{X}$*

Proof.
We have to show that for any set A belonging to $\mathcal{X}$, $\mathrm{Pred}(A)$ is well ordered for $\subset$.
Consider two sets $B_1 = \mathrm{Ball}(p,\lambda)$ and $B_2 = \mathrm{Ball}(q,\mu)$ containing both A. We have to show that they are comparable for $\subset$.
Let r be a point of A. This point belongs to both $\mathrm{Ball}(p,\lambda)$ and $\mathrm{Ball}(q,\mu)$, hence it is centre of each of these balls : $\mathrm{Ball}(p,\lambda) = \mathrm{Ball}(r,\lambda)$ and $\mathrm{Ball}(q,\mu) = \mathrm{Ball}(r,\mu)$. If $\lambda = \mu$, then $\mathrm{Ball}(p,\lambda)$ and $\mathrm{Ball}(q,\mu)$ are identical. If $\lambda < \mu$, then $\mathrm{Ball}(p,\lambda) = \mathrm{Ball}(q,\mu)$. This establishes that $\mathrm{Pred}(A)$ is well ordered for $\subset$ ∎

Proposition 12 *We have a one to one correspondance between partial ultrametric distances χ and stratified dendrograms $\mathcal{X}$*

4 Partial partitions

Consider a dendrogram Π verifying : $A \in \mathrm{supp}(\Pi) \Rightarrow \mathrm{Pred}(A) = A$. Let us verify that such a dendrogram is a partial partition, as they have been called by Ch. Ronse in [5].

Its partial ultrametric distance π is now a binary, taking its values in $\{0, 1\}$. It verifies

- (pp1) : for $p, q, r \in \text{supp}(\Pi)$: $\pi(p, q) = \pi(q, p) \leq \pi(q, r) \vee \pi(r, p)$: symmetry and ultrametric inequality
- (pp2) for $p \notin \text{supp}(\Pi), \forall q \in E : \pi(p, q) = 1$

This last relation is also true for p itself : for $p \notin \text{supp}(\Pi) : \pi(p, p) = 1$

The domain $\text{supp}(\Pi) = \{p \in E : \pi(p, p) = 0\}$ is called support of the partial partition. If this domain equals E, then π is a partition. Otherwise π is a partial partition.

Consider now a point $p \notin \text{supp}(\Pi)$. We call such points "aliens". For any $q \in E$, we have $1 = \pi(p, p) \leq \pi(p, q) \vee \pi(q, p) = \pi(p, q)$, showing that the ultrametric distance between an alien and any other point is 1.

Remark 13 *Aliens should not be mixed up with the singletons, which duly belong to the support. The singleton $\{x\}$ is a set of $\mathcal{P}(E)$ reduced to the point x. Singletons are characterized by: $\forall q \in E\ ,\ p \neq q, : \pi(p, q) = 1$ and $\pi(p, p) = 0$.*

We call $\text{cl}(p)$ the closed ball of centre p and of radius 0 associated to π. Relation (pp2) implies that for $p \notin \text{supp}(\Pi)$ the class $\text{cl}(p)$ is empty.

Consider now $p, q \in E$ such that $q \in \text{cl}(p)$. This shows that $p, q \in \text{supp}(\Pi)$ and $\pi(p, q) = 0$. If $r \in \text{cl}(p)$; then $\pi(p, r) = 0$ and $\pi(q, r) \leq \pi(q, p) \vee \pi(p, r) = 0$ showing that $r \in \text{cl}(q)$. Similarly $r \in \text{cl}(q) \Rightarrow r \in \text{cl}(p)$.

Hence for any $p, q \in E\ ,\ q \in \text{cl}(p) \Rightarrow \text{cl}(q) = \text{cl}(p)$.

But these are precisely the criteria given by Ch. Ronse for defining partial partitions:

- (P1b) for any $p \in E\ ,\ \text{cl}(p) = \varnothing$ or $p \in \text{cl}(p)$
- (P2a) for any $p, q \in E\ ,\ q \in \text{cl}(p) \Rightarrow \text{cl}(q) = \text{cl}(p)$

4.0.1 Partial equivalence relations

Associated to π, we may define a partial equivalence relation defined by $p\ R\ q \Leftrightarrow \pi(p, q) = 0$, which is symmetric and transitive but not reflexive. The support of a partial equivalence relation R is precisely $\text{supp}(\Pi)$; it also is the set of all points $p \in E$ for which there exists a point $q \in E$ verifying $p\ R\ q$. Ch. Ronse introduced this partial equivalence in [5].

5 Order relation between hierarchies and partial hierarchies

Let $\mathcal{A}$ and $\mathcal{B}$ be two dendrograms with their associated PUD : $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$. The following relation defines an order relation between the hierarchies: $B \leq A \Leftrightarrow \forall p, q \in E \quad \chi_{\mathcal{A}}(p, q) \leq \chi_{\mathcal{B}}(p, q)$

It follows that $\forall p \in E : \mathrm{Ball}_{\mathcal{B}}(p, \rho) \subset \mathrm{Ball}_{\mathcal{A}}(p, \rho)$. We say that the hierarchy $\mathcal{A}$ is coarser than the hierarchy $\mathcal{B}$ and that the hierarchy $\mathcal{B}$ is finer than the hierarchy $\mathcal{B}$.

For each $p \notin \mathrm{supp}(\mathcal{A}) : \chi_{\mathcal{A}}(p,p) = L$ which implies that $\chi_B(p,p) = L$, indicating that $\overline{\mathrm{supp}(\mathcal{A})} \subset \overline{\mathrm{supp}(\mathcal{B})}$, or equivalently $\mathrm{supp}(\mathcal{B}) \subset \mathrm{supp}(\mathcal{A})$

The smallest partial hierarchy has an empty support and contains only aliens, i.e. points p verifying $\forall q \in E\ ,\ \chi(p,q) = L$. The largest hierarchy is E itself, whose PUD verifies: $\forall p, q \in E : \chi\,(p,q) = 0$.

In the case where there are no aliens, that is $\mathrm{supp}(\mathcal{A}) = E$, then the largest hierarchy verifies $\chi(p,q) = L$ for $p \neq q$, and $\chi(p,p) < L$. It contains only singletons. If the stratification index of the singletons is 0 then $\chi(p,p) = 0$.

To binary PUDs $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$ correspond partitions and partial partitions. Their closed balls verify : $\mathrm{Ball}_{\mathcal{B}}(p, 0) \subset \mathrm{Ball}_{\mathcal{A}}(p, 0)$, the aliens remaining outside the balls. Hence the tiles of the finer partition $\mathcal{B}$ are included in the tiles of the coarser partition $\mathcal{A}$ which is coherent with the usual definition of the order between partitions.

5.1 Partial partitions by thresholding partial hierarchies

Summary : Increasing thresholds of a partial hierarchy produce increasing partial partitions. Inversely to a series of increasing partial partitions may be associated partial hierarchies which differ only by their stratification index.

Decomposition into partial partitions of a partial hierarchy Consider a partial hierarchy $\mathcal{X}$ with its associated PUD χ. By thresholding the PUD at level λ one obtains a partial binary ultrametric half distance (PBUD):

$\pi_\lambda(x,y) = \begin{array}{l} 1 \text{ if } \chi(x,y) > \lambda \\ 0 \text{ if } \chi(x,y) \leq \lambda \end{array}$ associated to a partial partition Π_λ.

For increasing thresholds λ, the series of PUD is decreasing and the associated partial partitions increasing, i.e. coarser:

$\lambda \geq \mu \Rightarrow \pi_\lambda \leq \pi_\mu \Rightarrow \Pi_\lambda \geq \Pi_\mu$

It is easy to verify that if χ is a PUD, then each π_λ also is a PUD. Let us check the ultrametric inequality.

For $p, q, r \in E : \chi(p,r) \leq \chi(p,q) \vee \chi(q,r)$.

If $\chi(p,r) \leq \lambda$, then $\pi_\lambda(p,r) = 0 \leq \pi_\lambda(p,q) \vee \pi_\lambda(q,r)$

If $\chi(p,r) > \lambda$, then $\pi_\lambda(p,r) = 1$ and $\chi(p,q) \vee \chi(q,r) > \lambda$, implying that $\chi(p,q) > \lambda$ or $\chi(q,r) > \lambda$, hence $\pi_\lambda(p,q) = 1$ or $\pi_\lambda(q,r) = 1$

5.2 Reconstructing a hierarchy from nested partial partitions

Inversely, to a family $(\Pi_i)_{i \in I}$ of increasing partial partitions may be associated a series of partial hierarchies sharing the same undelying dendrogram but with stratification indices which differ one from each other by an increasing anamorphosis. Let π_i the PUD associated to Π_i.

If ψ is an increasing anamorphosis, the partial hierarchy associated to $(\Pi_i)_{i\in I}$ and ψ has for PUD : $\chi_\psi = \sum \psi(i)\pi_i$.

5.3 Aliens and singletons

Both aliens and singletons play a particular role in the half distance, as they play no role in the the ultrametric inequality. The relation : $p, q \in E : \chi(p,q) \leq \chi(p,p) \vee \chi(p,q)$ is true for any value of $\chi(p,p)$

On the other hand, the distance between an alien and any other of its neighbors is at least as high that its own stratification index. For any $q \in E$, we have $\lambda = \chi(p,p) \leq \chi(p,q) \vee \chi(q,p) = \chi(p,q)$, showing that the ultrametric distance between p and any other point is larger than or equal to λ.

5.3.1 Partial hierarchies

Consider now a hierarchy $\mathcal{X}$ with its PUD χ and its thresholds π_λ at level λ. We define its open and closed supports at level λ.

Its open support at level λ is $\overset{\circ}{\mathrm{supp}}_\lambda(\chi) = \{p \mid \chi(p,p) < \lambda\}$ and its closed support $\overline{\mathrm{supp}}_\lambda(\chi) = \{p \mid \chi(p,p) \leq \lambda\}$.

For increasing values of λ, the partial partitions π_λ obtained by thresholding χ have increasing supports $\overline{\mathrm{supp}}_\lambda(\chi) = \{p \in E : \chi(p,p) \leq \lambda\}$. This means that a point p may be outside the support of partition π_λ and inside the support of partition π_μ for $\mu > \lambda$.

Similarly, as the partitions π_λ become coarser for increasing λ, a point p may be a singleton up to level λ and not a singleton anymore for higher levels. Let us analyse more precisely this behaviour.

Let $p \in E$ be a point verifying $\chi(p,p) = \lambda$ and the partition π_μ obtained by thresholding χ at the level μ. And consider $\nu = \bigwedge_{q \neq p} \chi(p,q)$. Since $\chi(p,p) \leq \chi(p,q) \vee \chi(q,p)$, we have $\nu \geq \lambda$. We are now to analyse the role played by p in the partition π_μ for increasing values of μ :

- case 1: $\mu < \lambda : \pi_\mu(p,p) = 1$ and p is an alien
- case 2: $\lambda \leq \mu < \nu : \pi_\mu(p,p) = 0$ but for $q \neq p$, $\pi_\mu(p,q) = 1$ and p is a singleton
- case 3: $\nu \leq \mu : \pi_\mu(p,p) = 0$ and there exists $q \neq p$ such that $\pi_\mu(p,q) = 0$ showing that p is a regular node and not a singleton.

Particular cases

- a "forever singleton" is a singleton for all π_μ. It is characterized by $p \in E : \chi(p,p) = 0$ and $\forall q \in E\ ,\ p \neq q : \chi(p,q) = L$. In this case $\lambda = 0$ and $\nu = L$
- a "singleton from λ on" is a singleton for all π_μ for $\mu \geq \lambda$. It is characterized by $p \in E : \chi(p,p) = \lambda$ and $\forall q \in E\ ,\ p \neq q : \chi(p,q) = L$. In this case $\nu = L$

- a "never singleton" : if $\chi(p,p) = \bigwedge_{q \neq p} \chi(p,q)$, then case 2 does not happen and p is never a singleton
- a "never alien" is characterised by $p \in E : \chi(p,p) = 0$
- a "forever alien" is an alien for all π_λ. It is characterized by $p \in E$: $\chi(p,p) = L$. In this case $\lambda = \nu = L$

6 Pruning of dendrograms

Consider now a grain opening ψ and a dendrogram $\mathcal{X}$. A grain opening applied on a set A takes it or leaves it : $\psi(A) \in \{A, \varnothing\}$

As any opening, ψ is anti-extensive, increasing and idempotent.

Proposition 14 *If we apply ψ to each $A \in \mathcal{X}$ we obtain a family $\psi(A)$ which again is a dendrogram written $\psi(\mathcal{X})$.*

Proof.
Consider a set $B \in \psi(\mathcal{X})$. There exists a set $A \in \mathcal{X}$ such that $B = \psi(A)$. As $\mathcal{X}$ is a dendrogram, $\mathrm{Pred}(A)$ is completely ordered for $\subset$. As ψ is anti-extensive and increasing, $\mathrm{Pred}(\psi A)$ is obtained by applying ψ to each set X belonging to $\mathrm{Pred}(A)$ yielding $\psi(X) = X$. Hence $\mathrm{Pred}(\psi A)$ also is completely ordered for $\subset$.
■

See [8] for an overview of the various pruning strategies of mintrees.

7 Partitions and Hierarchies

7.1 Definition of hierarchies

Fig.1 presents a hierarchy whereas fig.2 presents a dendrogram, presenting various lakes flooding a topographic surface. Areas which are covered by at least one lake belong to the support of the dendrogram. However, a lake C which contains a smaller lower lake C' is not a leave of the dendrogram. The points of C which do not belong to C' belong to the support of the dendrogram but do not belong to any leave. On the contrary fig.1 also presents a dendrogram, with the same order relation between the regions, but each point of the support of the dendrogram belongs to a leave.

Definition 15 *We call hierarchy $\mathcal{H}$ a dendrogram verifying:* $\bigcup \mathrm{Leav}(\mathcal{H}) = \mathrm{supp}(\mathcal{H})$

Proposition 16 *A dendrogram $\mathcal{X}$ is a hierarchy if and only if it verifies the union axiom:*
*(**Union axiom**) Any element A of $\mathcal{X}$ is the union of all other elements of $\mathcal{X}$ contained in A:*
$\forall A \in \mathcal{X} : \bigcup \{B \in \mathcal{X} \mid B \subset A \,; B \neq A\} = \{A, \emptyset\}$

Proof.
a) Suppose $\bigcup \mathrm{Leav}(\mathcal{X}) = \mathrm{supp}(\mathcal{X})$
Consider a set $A \in \mathcal{X}$ and a point $p \in A \subset \mathrm{supp}(\mathcal{X})$. There exists $B \in \mathrm{Leav}(\mathcal{X})$ and $x \in B$. If $B = A$ then $\bigcup \{B \in \mathcal{X} \mid B \subset A\ ; B \neq A\} = \{\emptyset\}$. On the other hand if $B \neq A$ then A and B have a non empty intersection, implying $A \subset B$ or $B \subset A$. As B is a leave, we necessarily have $B \subset A$. This shows that $\bigcup \{B \in \mathcal{X} \mid B \subset A\ ; B \neq A\} = \{A\}$
b) Suppose now that $\forall A \in \mathcal{X} : \bigcup \{B \in \mathcal{X} \mid B \subset A\ ; B \neq A\} = \{A, \emptyset\}$
It is always true that $\bigcup\limits_{p \in \mathrm{supp}(\mathcal{X})} \mathrm{home}(p) = \mathrm{supp}(\mathcal{X})$
Let us show that for any $p \in \mathrm{supp}(\mathcal{X})$, we have $A = \mathrm{home}(p) \in \mathrm{Leav}(\mathcal{X})$
If $\mathrm{home}(p) \notin \mathrm{Leav}(\mathcal{X})$, then $\mathrm{home}(p)$ contains sets $B \in \mathcal{X}, B \neq A$ and $p \notin B$
Hence $p \notin \bigcup \{B \in \mathcal{X} \mid B \subset A\ ; B \neq A\}$ showing that the hypothesis $\bigcup \{B \in \mathcal{X} \mid B \subset A\ ; B \neq A\} = \{A, \emptyset\}$ is false.
Hence $\mathrm{home}(p) \in \mathrm{Leav}(\mathcal{X})$, and $\bigcup\limits_{p \in \mathrm{supp}(\mathcal{X})} \mathrm{home}(p) = \bigcup \mathrm{Leav}(\mathcal{X}) = \mathrm{supp}(\mathcal{X})$. ■

7.2 Partitions and partial partitions

All partial partitions and partitions verify the union axiom.

Proposition 17 *Partial partitions are hierachies.*

Proof.
A partial partition Π has been defined as a dendrogram for which each $A \in \Pi$ verifies $\mathrm{Pred}(A) = \{A\}$. But then each $A \in \Pi$ is a leave and the support of Π is the union of the leaves of Π. ■

If furthermore $\mathrm{supp}(\Pi) = E$, then the partial partition becomes a partition.

Hierarchies which verify $\mathrm{supp}(\mathcal{H}) = E$ are called covering hierarchies, since each point of E belongs to a leave of $\mathcal{H}$.

7.3 Covering hierarchies

Consider a covering hierarchy $\mathcal{H}$ with a stratification index st and the derived partial ultrametric distance η.

Let $\mathrm{supp}(\mathcal{H}) = E$, then for all $p \in E : \eta(p,p) < L$

If furthermore we impose that the stratification index of all leaves is equal to 0, then for all $p \in E : \eta(p,p) = 0$. In such a situation η is a half distance, as defined by L. Schwartz in [9]. His definition of half-distances and half-metric spaces is given below. Furthermore the open support $\overset{\circ}{\mathrm{supp}}_\lambda(\chi) = \{p \mid \chi(p,p) < \lambda\}$ and closed support $\overline{\mathrm{supp}}_\lambda(\chi) = \{p \mid \chi(p,p) \leq \lambda\}$ are invariant with λ and are equal to $\mathrm{supp}(\mathcal{H})$.

7.4 Half distances and half metric spaces

Definition 18 *A half-distance on a domain E is a mapping d from $E \times E$ into $\mathbb{R}^+$ with the following properties:*

1) Symmetry : $d(x, y) = d(y, x)$
2) Half-positivity: $d(x, y) \geq 0$ and $d(x, x) = 0$
3) Triangular inequality: $d(x, z) \leq d(x, y) + d(y, z)$

Definition 19 *A half metric space is a set E with a family $(d_i)_{i \in I}$ of half-distances verifying the following condition:*
the family $(d_i)_{i \in I}$ is a directed set, i.e. for any finite subset J of I, there exists an index $k \in I$ such that $d_k \geq d_j$ for all $j \in J$

The open half balls $B_{i,o}(a, R)$ (resp. closed $B_i(a, R)$ of a center $a \in E$, of radius R and index i are the sets of all x of E such that $d_i(a, x) < R$ (resp. $\leq R$).

A half metric space is then a topological space defined as follows : a subset $\mathcal{O}$ of E is open, if for each point $x \in \mathcal{O}$, there exists a half ball $B_i(x, R)$ centered at x, with a positive radius entirely contained in $\mathcal{O}$.

If the triangular inequality is replaced by the ultrametric inequality, we call it ultrametric half-distance or ultrametric ecart. To any hierarchy $\mathcal{X}$ defined on subsets of $\mathcal{P}(E)$ is thus associated an ultrametric half-distance or ultrametric ecart.

7.5 Exchanging aliens and singletons

Singletons play a particular role in the work of J.Serra and Ch. Ronse on partitions and partial partitions. For instances, for eroding a partition, J.Serra proposes to erode each tile separately and to fill the empty spaces with singletons [11]. Ch. Ronse proposes an operator RS, removing the singletons and transforming a partition into a partial partition [7]. The same operation becomes simpler by using aliens : singletons are simply turned into aliens.

In this section we define two operators exchanging aliens and singletons in dendrograms. These operators form an adjunction. Let ζ and χ be the PUD of two dendrograms (partial hierarchies) $\mathcal{Z}$ and $\mathcal{X}$.

7.5.1 An operator s2a transforming all singletons of $\mathcal{Z}$ into "never singletons"

We define an operator s2a which transforms all singletons of $\mathcal{Z}$ into "never singletons". The PUD of s2a$(\mathcal{Z})$ is defined by:

For $p \in E$, $\mathrm{s2a}\,\zeta(p, p) = \bigwedge_{q \neq p} \zeta(p, q) = \mu$

For $p, q \in E, p \neq q : \mathrm{s2a}\,\zeta(p, q) = \zeta(p, q)$

For any threshold $\lambda < \mu$ of $\mathrm{s2a}\,\zeta(p, p)$, we get a partial partition for which p is an alien, i.e is outside the support. And for any threshold $\lambda \geq \mu$, p in incorporated in the support without becoming a singleton, since it belongs to a region with at least two points.

It is easy to check that if ζ is a PUD, then $\mathrm{s2a}\,\zeta$ still is a PUD. As a matter of fact, ζ and $\mathrm{s2a}\,\zeta$ differ only on the singletons and aliens, which, following the remark made earlier, play no role in the propagation of the half distances through the ultrametric inequality.

Remark 20 *In the binary case, if p is a singleton, for $p \neq q : \zeta(p,q) = 1$ and* $\mathrm{s2a}\,\zeta(p,p)$*, showing that p becomes an alien and exits the support.*

7.5.2 An operator a2s transforming all aliens of $\mathcal{X}$ into "never aliens"

The operator a2s simply transforms all aliens of $\mathcal{X}$ into "never aliens" by doing: for $p \in E$: $\mathrm{a2s}\,\chi(p,p) = 0$. Obviously the operator a2s transforms a partial hierarchy into a covering hierarchy, by adding singletons outside its support.

The following lemma is the counterpart of Lemma8.2 in [7].

Lemma 21 *If $\mathcal{X}$ is a hierarchy and $\mathcal{X}'$ a partial hierarchy and χ,χ' their associated PUDs, then* $\mathrm{a2s}\,\mathcal{X}' \leq \mathcal{X} \Rightarrow \mathcal{X}' \leq \mathcal{X}$

Proof. We suppose $\mathrm{a2s}\,\mathcal{X}' \leq \mathcal{X}$, i.e. $\chi \leq \mathrm{a2s}\,\chi'$. But $\mathrm{a2s}\,\chi' \leq \chi'$, after transforming the aliens into "never aliens". Hence $\chi \leq \mathrm{a2s}\,\chi' \leq \chi' \Rightarrow \mathcal{X}' \geq \mathcal{X}$
■

7.5.3 The operators a2s and s2a form an adjunction

We have to prove that for any pair ζ and χ of PUDs, we have : $\mathrm{a2s}\,\chi \leq \zeta \Leftrightarrow \chi \leq \mathrm{s2a}\,\zeta$. The operator s2a is then an erosion on the PUDs and a dilation on the corresponding partial hierarchy. And the operator a2s is then a dilation on the PUD and an erosion on the corresponding partial hierarchy.
Proof.
We first consider the case $p, q \in E, p \neq q$:
$\mathrm{a2s}\,\chi(p,q) = \chi(p,q)$ and $\zeta(p,q) = \mathrm{s2a}\,\zeta(p,q)$
Hence $\mathrm{a2s}\,\chi(p,q) \leq \zeta(p,q) \Leftrightarrow \chi(p,q) \leq \mathrm{s2a}\,\zeta(p,q)$
Consider now the pair (p,p) :
1) Suppose $\mathrm{a2s}\,\chi \leq \zeta$
$\mathrm{s2a}\,\zeta(p,p) = \bigwedge_{q \neq p} \zeta(p,q)$ but $\mathrm{a2s}\,\chi \leq \zeta$ hence $\mathrm{s2a}\,\zeta(p,p) = \bigwedge_{q \neq p} \zeta(p,q) \geq \bigwedge_{q \neq p} \mathrm{a2s}\,\chi(p,q)$
And for $p, q \in E, p \neq q$: $\mathrm{a2s}\,\chi(p,q) = \chi(p,q)$ hence $\bigwedge_{q \neq p} \mathrm{a2s}\,\chi(p,q) = \bigwedge_{q \neq p} \chi(p,q) \geq \chi(p,p)$ the last inequality deriving from the ultrametric inequality $\chi(p,p) \leq \chi(p,q) \vee \chi(q,p)$
Hence $\mathrm{s2a}\,\zeta(p,p) = \bigwedge_{q \neq p} \zeta(p,q) \geq \bigwedge_{q \neq p} \mathrm{a2s}\,\chi(p,q) = \bigwedge_{q \neq p} \chi(p,q) \geq \chi(p,p)$
2) Suppose $\chi \leq \mathrm{s2a}\,\zeta$
For all $p \in E$: $\mathrm{a2s}\,\chi(p,p) = 0$ and $\zeta(p,p) \geq 0$, hence $\mathrm{a2s}\,\chi(p,p) \leq \zeta(p,p)$
Finally we have proved that $\mathrm{a2s}\,\chi \leq \zeta \Leftarrow \chi \leq \mathrm{s2a}\,\zeta$ ■

7.5.4 Discussion

The couple $(\mathrm{s2a}, \mathrm{a2s})$ is an adjunction on the PUDs, $(\mathrm{a2s}, \mathrm{s2a})$ on the corresponding partial hierarchies. The operator s2a is an erosion on the PUDs and a dilation on the corresponding partial hierarchies. And the operator a2s is then a dilation on the PUD and an erosion on the corresponding partial hierarchies. This last erosion transforms any partial hierarchy into a hierarchy.

In the binary case, where the PUD take only values 0 and 1, we get the operators already introduced by Ch. Ronse in [7]. The operator s2a transforms all singletons into aliens, in other words it removes all singleton blocks from the partial partition. Hence it is identical with the operator RS introduced by Ch. Ronse in [7]. Similarly the operator a2s fills a partial partition by singleton blocks outside its support, hence it is identical with the operator FS in [7].

Theorem 9 in [7] establishes a number of properties of FS and RS. As we take s2a and a2s as operators between partial hierarchies, we do not have to consider the operator IN and the results become simpler.

Here are some properties which are equivalent with properties of theorem 9, transposed in our framework.

- the operator a2s is increasing and anti-extensive on PUD : it is an opening on PUDs and a closing on the corresponding partial hierarchies.

- the operator s2a is increasing and extensive on PUD : it is a closing on PUDs and an opening on the corresponding partial hierarchies.

- a2s(s2a) = a2s , showing again that a2s is an opening and s2a(a2s) = s2a is a closing on PUD.

7.5.5 An adjunction based on aliens and singletons of high rank

In some cases, one wants to apply the preceding operators only to singletons and aliens with a stratification level higher than a given value λ.

The operator s2a_λ is defined by : for $p, q \in E$, $\text{s2a}_\lambda\, \zeta(p,p) = \bigwedge\limits_{q \neq p} \zeta(p,q)$ if $\zeta(p,p) \geq \lambda$ and $\text{s2a}_\lambda\, \zeta(p,p) = \zeta(p,p)$ otherwise. This operator transforms only singletons with a stratification level higher than λ into "never singletons".

The operator a2s_λ is defined by : for $p, q \in E$, $\text{a2s}_\lambda\, \chi(p,p) = \chi(p,p) \wedge \lambda$. It transforms aliens with a stratification level higher than λ into "never aliens above λ".

The couple $(\text{s2a}_\lambda, \text{a2s}_\lambda)$ also is an adjunction on the PUDs.

7.6 The blending and grinding operators

In this section, we extend to partial hierarchies the blending and grinding operators presented by Ch. Ronse p.358-360 of [7]

7.6.1 The $\lambda - identity$ hierarchy of A and the $\lambda - universal$ hierarchy of A.

For $A \in \mathcal{P}(E)$, we define the $\lambda - identity$ hierarchy of A (we write $\mathbf{0}^\lambda_\bigstar(A)$) through its PUHD $\mathbf{0}^\lambda_A$ by the following relations :

- for $p \in A : \mathbf{0}_A^\lambda(p,p) = \lambda$
- for $p \notin A : \mathbf{0}_A^\lambda(p,p) = L$
- for $p \neq q \in E : \mathbf{0}_A^\lambda(p,q) = L$

Interpretation of the partial hierarchy $\mathbf{0}_\bigstar^\lambda(A)$:

- Inside A : up to level λ, $\mathbf{0}_\bigstar^\lambda(A)$ has only aliens and above λ only singletons.
- Outside A, it has aliens at all levels.

In the binary case, $\lambda = 0$ and $L = 1$ and $\mathbf{0}_A^\lambda$ is identical with $\mathbf{0}_A$ defined by Ch. Ronse p.354, which partitions A into its singletons, and fills the complement of A with aliens.

For $A \in \mathcal{P}(E)$, we define the $\lambda - universal$ hierarchy of A (we write $\mathbf{1}_\bigstar^\lambda(A)$) through its PUHD $\mathbf{1}_A^\lambda$ by the following relations :

- for $p, q \in A : \mathbf{1}_A^\lambda(p,q) = 0$
- for $p \notin A$ or $q \notin A : \mathbf{1}_A^\lambda(p,q) = \lambda$

Interpretation: For all levels up to λ, the partial hierarchy $\mathbf{1}_\bigstar^\lambda(A)$ has one block identical with A and aliens outside A. For the levels above λ, it has one block identical with E. In the binary case, $\lambda = 1$ and $\mathbf{1}_\bigstar^\lambda(A)$ is identical with $\mathbf{1}_A$ defined by Ch. Ronse p.354, representing a partial partition with only one block equal to A.

We now consider two mappings from $\mathcal{P}(E)$ into the PUHDs of partial hierarchies :

- $\mathbf{1}_\bigstar^\lambda : A \rightarrowtail \mathbf{1}_A^\lambda$
- $\mathbf{0}_\bigstar^\lambda : A \rightarrowtail \mathbf{0}_A^\lambda$

Notation: We write $\mathbf{1}_\bigstar^\lambda(A)$ (resp. $\mathbf{0}_\bigstar^\lambda(A)$) for the partial hierarchy associated to the PUHD $\mathbf{1}_A^\lambda$ (resp. $\mathbf{0}_A^\lambda$)

7.6.2 The open and closed $\lambda - supports$ of a partial hierarchy

Consider now a hierarchy $\mathcal{X}$ with its PUHD χ. We define its open and closed supports at level λ.

Its open support at level λ is $\overset{\circ}{\mathrm{supp}}_\lambda(\chi) = \{p \mid \chi(p,p) < \lambda\}$ and its closed support $\overline{\mathrm{supp}}_\lambda(\chi) = \{p \mid \chi(p,p) \leq \lambda\}$

7.6.3 Two adjunctions between $\mathcal{P}(E)$ and the partial hierarchies

The adjunction $(\mathbf{0}^{\lambda}_{\bigstar}, \overline{\mathrm{supp}}_{\lambda})$ We have to show that for any $A \in \mathcal{P}(E)$ and any partial hierarchy $\mathcal{X}$, with a PUHD χ, we have :

$A \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi) \Leftrightarrow \mathbf{0}^{\lambda}_{\bigstar}(A) \leq \mathcal{X}$

We know that $\mathbf{0}^{\lambda}_{\bigstar}(A) \leq \mathcal{X} \Leftrightarrow \chi \leq \mathbf{0}^{\lambda}_{A}$, so we have to prove that $A \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi) \Leftrightarrow \chi \leq \mathbf{0}^{\lambda}_{A}$

Proof.

1) Suppose that $A \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi)$.

For $p \in A$, we have by definition $\mathbf{0}^{\lambda}_{A}(p,p) = \lambda$. As $A \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi)$ we also have $p \in \overline{\mathrm{supp}}_{\lambda}(\chi)$, hence $\chi(p,p) \leq \lambda$. It follows that for p, we have $\chi(p,p) \leq \mathbf{0}^{\lambda}_{A}(p,p)$.

Now for $p \notin A$: $\mathbf{0}^{\lambda}_{A}(p,p) = L$ and for $p \neq q \in E$: $\mathbf{0}^{\lambda}_{A}(p,p) = L$, so here also $\chi \leq \mathbf{0}^{\lambda}_{A}$

2) Supppose now $\chi \leq \mathbf{0}^{\lambda}_{A}$

Take $p \notin \overline{\mathrm{supp}}_{\lambda}(\chi)$, hence $\chi(p,p) > \lambda$. Hence $\mathbf{0}^{\lambda}_{A}(p,p) \geq \chi(p,p) > \lambda$ showing that $p \notin A$

It follows that $A \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi)$ ∎

The adjunction $(\overset{\circ}{\mathrm{supp}}_{\lambda}, \mathbf{1}^{\lambda}_{\bigstar})$ We have to show that for any $A \in \mathcal{P}(E)$ and any partial hierarchy $\mathcal{X}$, with a PUHD χ, we have :

$\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi) \subseteq A \Leftrightarrow \mathcal{X} \leq \mathbf{1}^{\lambda}_{\bigstar}(A)$

We know that $\mathcal{X} \leq \mathbf{1}^{\lambda}_{\bigstar}(A) \Leftrightarrow \chi \geq \mathbf{1}^{\lambda}_{A}$, so we have to prove that $\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi) \subseteq A \Leftrightarrow \chi \geq \mathbf{1}^{\lambda}_{A}$

Proof.

1) Suppose that $\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi) \subseteq A$.

For $p \notin \overset{\circ}{\mathrm{supp}}_{\lambda}(\chi)$, we have by definition $\chi(p,p) \geq \lambda$

For $p \notin \overset{\circ}{\mathrm{supp}}_{\lambda}(\chi)$ or $q \notin \overset{\circ}{\mathrm{supp}}_{\lambda}(\chi)$ and $p \neq q$ we also have $\chi(p,q) \geq \lambda$. But for any p, q we have $\lambda \geq \mathbf{1}^{\lambda}_{A}(p,q)$, it follows that $\chi(p,q) \geq \mathbf{1}^{\lambda}_{A}(p,q)$.

For $p, q \in \overset{\circ}{\mathrm{supp}}_{\lambda}(\chi)$ we have $\mathbf{1}^{\lambda}_{A}(p,q) = 0 \leq \chi(p,q)$

2) Suppose now $\chi \geq \mathbf{1}^{\lambda}_{A}$

Take p verifying $p \in \overset{\circ}{\mathrm{supp}}_{\lambda}(\chi)$; then $\chi(p,p) < \lambda$. And as then $\chi \geq \mathbf{1}^{\lambda}_{A}$ we have $\mathbf{1}^{\lambda}_{A}(p,p) < \lambda$, showing that $p \in A$. Hence $\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi) \subseteq A$ ∎

7.7 The blending and grinding operators

We now established $\mathbf{0}^{\lambda}_{\bigstar}(A) \leq \mathcal{X} \Leftrightarrow A \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi)$ and $\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi') \subseteq A \Leftrightarrow \mathcal{X}' \leq \mathbf{1}^{\lambda}_{\bigstar}(A)$

Replacing A by $\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi')$ in the first equivalence and by $\overline{\mathrm{supp}}_{\lambda}(\chi)$ in the second yields

$\mathbf{0}^{\lambda}_{\bigstar}(\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi')) \leq \mathcal{X} \Leftrightarrow \overset{\circ}{\mathrm{supp}}_{\lambda}(\chi') \subseteq \overline{\mathrm{supp}}_{\lambda}(\chi) \Leftrightarrow \mathcal{X}' \leq \mathbf{1}^{\lambda}_{\bigstar}(\overline{\mathrm{supp}}_{\lambda}(\chi))$, showing that $\left[\mathbf{1}^{\lambda}_{\bigstar}(\overline{\mathrm{supp}}_{\lambda}(\chi)), \mathbf{0}^{\lambda}_{\bigstar}(\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi))\right]$ also form an adjunction on the partial hierarchies, where $\mathbf{0}^{\lambda}_{\bigstar}(\overset{\circ}{\mathrm{supp}}_{\lambda}(\chi))$ is the dilation and $\mathbf{1}^{\lambda}_{\bigstar}(\overline{\mathrm{supp}}_{\lambda}(\chi)$ the erosion.

As $\mathbf{0}^{\lambda}_{\bigstar}(\overset{\circ}{\operatorname{supp}}_{\lambda}(\chi))$ is anti-extensive and idempotent, it is also an opening. And $\mathbf{1}^{\lambda}_{\bigstar}(\overline{\operatorname{supp}}_{\lambda}(\chi))$ is both an erosion and a closure.

7.7.1 Interpretation

Partial partitions The block blending closure and block grinding operator have been introduced by Ch. Ronse, p.359 in [7]. Partial partitions are particular partial hierarchies where the ultrametric 1/2 distance takes only the values 0 and 1. Let π be such a partial hierarchy. Its open support at level 1 is $\overset{\circ}{\operatorname{supp}}_1(\pi) = \{p \mid \pi(p,p) < 1\}$ and its closed support at level 0 is $\operatorname{supp}_0(\pi) = \{p \mid \pi(p,p) \leq 0\}$ are identical with the support $\operatorname{supp}(\pi)$ of π.

- The block blending operator $\mathbf{1}_{\bigstar}(\operatorname{supp}(\pi))$ merges all blocks included in the support of π and produces aliens outside. It is also a closure.
- The block grinding operator $\mathbf{0}_{\bigstar}(\operatorname{supp}(\pi))$ pulverizes each block of the support of π into its singletons and produces aliens outside. It is also an opening.

Partial hierarchies

- The block blending operator $\mathbf{1}^{\lambda}_{\bigstar}(\overline{\operatorname{supp}}_{\lambda}(\chi))$produces a dendrogram with two regions. One region $\overline{\operatorname{supp}}_{\lambda}(\chi)$ with a stratification level equal to λ and one region equal to E with a stratification level greater than λ.
- For all levels $\mu \leq \lambda$, the block grinding operator $\mathbf{0}^{\lambda}_{\bigstar}(\overset{\circ}{\operatorname{supp}}_{\lambda}(\chi))$ produces a dendrogram where $\overset{\circ}{\operatorname{supp}}_{\lambda}(\chi)$ is pulverized in λ-singletons (points p verifying $\chi(p,p) = \lambda$), and outside $\overset{\circ}{\operatorname{supp}}_{\lambda}(\chi)$ there are only aliens verifying $\chi(p,p) > \lambda$.

8 The lattice of partial hierarchies.

It is often interesting to combine several hierarchies, in order to combine various criteria or merge the information obtained from diverse sources (colour or multispectral images for instance). We already defined an order relation between hierarchies. We show here how this order relation structures them into a complete lattice.

As a matter of fact, partial hierarchies and hierarchies have the same structure. The only difference lies in the supports. Hierarchies have the whole domain E as support, hence any combination of hierarchies keeps this same support. On the other hand, partial hierarchies do not occupy the whole domain E and one has to consider the domain of any combination of them.

In what follows we consider the general case of partial hierarchies.

8.1 Infimum of partial hierarchies

For the sake of simplicity and pedagogy we first consider the case of two hierarchies.

8.1.1 Case of two partial hierarchies

The infimum of two partial hierarchies $\mathcal{A}$ and $\mathcal{B}$ is written $\mathcal{A} \wedge \mathcal{B}$ and is defined by its ultrametric half-distance $\chi_{\mathcal{A}\wedge\mathcal{B}} = \chi_{\mathcal{A}} \vee \chi_{\mathcal{B}}$. It is easy to check that it is indeed a half-distance. It is symmtrical and half-positive. Let us check the ultrametric inequality:

$(\chi_{\mathcal{A}} \vee \chi_{\mathcal{B}})(p,r) \vee (\chi_{\mathcal{A}} \vee \chi_{\mathcal{B}})(r,q) = (\chi_{\mathcal{A}}(p,r) \vee \chi_{\mathcal{A}}(r,q)) \vee (\chi_{\mathcal{B}}(p,r) \vee \chi_{\mathcal{B}}(r,q)) > (\chi_{\mathcal{A}}(p,q) \vee \chi_{\mathcal{B}}(p,q)) = \chi_{\mathcal{A}} \vee \chi_{\mathcal{B}}(p,q)$

Its balls are defined by : $\forall p \in E : \text{Ball}_{\mathcal{A}\wedge\mathcal{B}}(p,\rho) = \text{Ball}_{\mathcal{A}}(p,\rho) \wedge \text{Ball}_{\mathcal{B}}(p,\rho)$.

The aliens of a partial hierarchy $\mathcal{X}$ are characterized by $\forall p,q \in E : \chi_{\mathcal{A}\wedge\mathcal{B}}(p,q) = \chi_{\mathcal{A}}(p,q) \vee \chi_{\mathcal{B}}(p,q) = \lambda > 0$. Hence $\chi_{\mathcal{A}}(p,q) = \lambda$ or $\chi_{\mathcal{B}}(p,q) = \lambda$, showing that $\left[\overset{\circ}{\text{supp}}_\lambda(\mathcal{A}\wedge\mathcal{B})\right]^C = \left[\overset{\circ}{\text{supp}}_\lambda(\mathcal{A})\right]^C \vee \left[\overset{\circ}{\text{supp}}_\lambda(\mathcal{B})\right]^C$ or equivalently $\overset{\circ}{\text{supp}}_\lambda(\mathcal{A}\wedge\mathcal{B})=\overset{\circ}{\text{supp}}_\lambda(\mathcal{A}) \wedge \overset{\circ}{\text{supp}}_\lambda(\mathcal{B})$. The aliens of $\mathcal{A}\wedge\mathcal{B}$ are the union of the aliens of $\mathcal{A}$ and of $\mathcal{B}$.

8.1.2 Infimum of a family of partial hierarchies

Consider now a family of hierarchies $(\mathcal{A}_i)_{i\in I}$, the PUHD of the hierarchy $\mathcal{A}_i$ being χ_i.

If this family is empty, its infimum is the greatest hierarchy $\mathcal{X}$, containing only one region, and whose PUHD verifies $\forall p,q \in E$, $\chi(p,q) = 0$.

For a non empty family, the PUHD of the infimum is defined by : $\chi_{\wedge\mathcal{A}_i} = \bigvee_i \chi_i$, the smallest PUHD larger or equal to each χ_i. And $\overset{\circ}{\text{supp}}_\lambda(\bigwedge \mathcal{A}_i) = \bigwedge_i \overset{\circ}{\text{supp}}_\lambda(\mathcal{A}_i)$.

8.2 Supremum of partial hierarchies

For the sake of simplicity and pedagogy here also we first consider the case of two hierarchies.

8.2.1 The subdominant ultrametric half-distance

The supremum of two hierarchies $\mathcal{A}$ and $\mathcal{B}$ is written $\mathcal{A}\vee\mathcal{B}$ and is the smallest hierarchy larger than $\mathcal{A}$ and $\mathcal{B}$.

As $\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}$ is not an ultrametric distance, we chose for $\chi_{\mathcal{A}\vee\mathcal{B}}$ the largest ultrametric distance which is lower than $\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}$. This distance exists: the set of ultrametric distances lower than $\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}$ is not empty, as the distance 0 is ultrametric ; furthermore, this family is closed by supremum, hence it has a largest element. Let us construct it.

Consider a series of points $(x_0, x_1, \cdots, x_n)$. As $\chi_{\mathcal{A}\vee\mathcal{B}}$ should be an ultrametric distance, we have for any path $x_0, x_1, ..., x_n$
$\chi_{\mathcal{A}\vee\mathcal{B}}((x_0, x_n) \le \chi_{\mathcal{A}\vee\mathcal{B}}(x_0, x_1) \vee \chi_{\mathcal{A}\vee\mathcal{B}}(x_1, x_2) \vee \cdots \vee \chi_{\mathcal{A}\vee\mathcal{B}}(x_{n-1}, x_n)$.
But for each pair of points x_i, x_{i+1} we have $\chi_{\mathcal{A}\vee\mathcal{B}}(x_i, x_{i+1}) \le [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_i, x_{i+1})$.
Hence $\chi_{\mathcal{A}\vee\mathcal{B}}(x_0, x_n) \le [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_0, x_1) \vee [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_1, x_2) \vee \cdots$
$\vee\, \chi\, [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_{n-1}, x_n)$.
There exists a chain along which the expression on the right becomes minimal and is equal to the maximal value taken by $[\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}]$ on two successive points of the chain. This maximal value is called sup section of the chain for $\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}$. For this reason, the chain itself is called chain of minimal sup-section. This valuation being an ultrametric ecart necessarily is the largest ultrametric ecart below $\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}$. Let us verify the ultrametric inequality.

For $p, q, r \in E$ there exists a chain between p and q along which $[\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](p, q)$ takes its value and another chain between q and r along which $[\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](q, r)$ takes its value. The concatenation of both chains forms a chain between p and q which is not necessarily the chain of lowest sup-section between them, hence:

$[\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](p, r) \le [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](p, q) \vee [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](q, r)$.

We write $\overbrace{\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}}$ for the subdominant ultrametric associated to $\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}$.

Let us analyse the supports. The support $\text{supp}_\lambda\, \mathcal{A} \vee \mathcal{B}$ are all points p verifying $\overbrace{\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}}(p, p) \le \lambda$. The aliens of $\mathcal{A} \vee \mathcal{B}$ verify $\overbrace{\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}}(p, p) > \lambda$. But

$\lambda < \overbrace{\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}}(p, p) \le \chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}(p, p)$, which implies that $\chi_{\mathcal{A}}(p, p) \ge \lambda$ and $\chi_{\mathcal{B}}(p, p) \ge \lambda$, hence $\left[\overset{\circ}{\text{supp}}_\lambda(\mathcal{A} \vee \mathcal{B})\right]^C \subset \left[\overset{\circ}{\text{supp}}_\lambda(\mathcal{A})\right]^C \wedge \left[\overset{\circ}{\text{supp}}_\lambda(\mathcal{B})\right]^C$ or equivalently $\overset{\circ}{\text{supp}}_\lambda(\mathcal{A} \vee \mathcal{B}) \supset \overset{\circ}{\text{supp}}_\lambda(\mathcal{A}) \vee \overset{\circ}{\text{supp}}_\lambda(\mathcal{B})$.

Geometrical interpretation Suppose that $(x_0, x_1, \cdots, x_n)$ is the chain for which $\overbrace{\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}}(x_0, x_n) = [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_0, x_1) \vee [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_1, x_2) \vee \cdots$
$\vee [\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_{n-1}, x_n)$ is minimal with a value λ. Then $[\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_i, x_{i+1}) \le \lambda$ means that the ball $\text{Ball}_{\mathcal{A}}(x_i, \lambda)$ or the ball $\text{Ball}_{\mathcal{B}}(x_i, \lambda)$ contains the point x_{i+1}. If it is $\text{Ball}_{\mathcal{A}}(x_i, \lambda)$, then x_{i+1} also is center of this ball. Hence a series of points $x_k, x_{k+1}, x_{k+2,...}$ all belong to the same ball $\text{Ball}_{\mathcal{A}}(x_i, \lambda)$, they are all centers of this ball and it is possible to keep only one of them and suppress all others from the list. Like that we get a path where the firsts two points x_0, x_1 belong to one of the balls, say $\text{Ball}_{\mathcal{A}}(x_0, \lambda)$, the couple x_1, x_2 belong to the other $\text{Ball}_{\mathcal{B}}(x_2, \lambda)$, and so on. The successive ovelapping pairs of points belong alternatively to balls $\text{Ball}_{\mathcal{A}}$ or $\text{Ball}_{\mathcal{B}}$.

Since $[\chi_{\mathcal{A}} \wedge \chi_{\mathcal{B}}](x_i, x_{i+1}) \le \lambda$, both points x_i and x_{i+1} are within the closed support $\overline{\text{supp}}_\lambda(\chi_{\mathcal{A}}) \wedge \overline{\text{supp}}_\lambda(\chi_{\mathcal{B}})$ (recall that $\overline{\text{supp}}_\lambda(\chi) = \{p \mid \chi(p, p) \le \lambda\}$)

The necessity of chaining blocks for obtaining suprema of partitions is well known [10] ; Ronse has confirmed that it is still the case for partial partitions [7].

8.2.2 Supremum of a family of hierarchies or partial hierarchies

Consider now a family of hierarchies $(\mathcal{A}_i)_{i\in I}$, the PUHD of the hierarchy $\mathcal{A}_i$ being χ_i.

If this family is empty, its supremum is the smallest hierarchy $\mathcal{X}$. In the case of hierarchies, this smallest hierarchy only contains singletons, whose PUHD verifies $\forall p \neq q \in E,\ \chi(p,q) = L$, and $\forall p \in E,\ \chi(p,q) = 0$. The smallest hierarchy among the partial hierarchies contains only aliens and its support is empty ; its PUHD verifies $\forall p,q \in E,\ \chi(p,q) = L$.

For a non empty family, the PUHD of the supremum is defined by : $\chi_{\vee \mathcal{A}_i} = \overbrace{\bigwedge_i \chi_i}$, that is the subdominant partial ultrametric distance associated to the family $(\mathcal{A}_i)_{i\in I}$,.the largest ultrametric distance which is lower than $\bigwedge_i \chi_i$. This distance exists: the set of ultrametric distances lower than $\bigwedge_i \chi_i$ is not empty, as it contains the largest hierarchy, containing only one region, and whose PUHD verifies $\forall p,q \in E,\ \chi(p,q) = 0$. Furthermore, this family is closed by supremum, hence it has a largest element.

Its expression may be found in a similar manner as in the case of only two hierarchies. Consider a series of points $(x_0, x_1, \cdots, x_n)$. As $\chi_{\vee \mathcal{A}_i}$ should be an ultrametric distance, we have for any path $x_0, x_1, ..., x_n$
$\chi_{\vee \mathcal{A}_i}((x_0, x_n) \leq \chi_{\vee \mathcal{A}_i}(x_0, x_1) \vee \chi_{\vee \mathcal{A}_i}(x_1, x_2) \vee \cdots \vee \chi_{\vee \mathcal{A}_i}(x_{n-1}, x_n)$.
But for each pair of points x_i, x_{i+1} we have $\chi_{\vee \mathcal{A}_i}(x_i, x_{i+1}) \leq [\bigwedge_i \chi_i](x_i, x_{i+1})$.
Hence $\chi_{\vee \mathcal{A}_i}(x_0, x_n) \leq [\bigwedge_i \chi_i](x_0, x_1) \vee [\bigwedge_i \chi_i](x_1, x_2) \vee \cdots \vee \chi[\bigwedge_i \chi_i](x_{n-1}, x_n)$.
There exists a chain along which the expression on the right becomes minimal and is equal to the maximal value taken by $[\bigwedge_i \chi_i]$ on two successive points of the chain. This maximal value is called sup section of the chain for $\bigwedge_i \chi_i$. For this reason, the chain itself is called chain of minimal sup-section. This valuation being an ultrametric ecart necessarily is the largest ultrametric ecart below $\bigwedge_i \chi_i$.

The expression of $\chi_{\vee \mathcal{A}_i}(p,q) = \inf_{p=x_0, x_n=q} \bigvee_{k=0}^{n-1} \bigwedge_i \chi_i(x_k, x_{k+1})$.

Nota bene: The infimum $\inf_{p=x_0, x_n=q}$ has to be taken on all chains of points between p and q.

8.2.3 Illustration

Fig.4 presents too hierarchies HA and HB through their nested partitions. The supremum and infimum of both hierarchies also are represented. The infimum takes for each threshold the intersection of the corresponding partitions, obtained through intersection of the tiles. The supremum is obtained by keeping only the boundaries existing in each component.

Fig.5 presents an initial image to segment.The H component and the V component of the colour image are segmented separately, yielding two hierarchies. Each hierarchy is illustrated trough one of its thresholds. The infimum of both hierarchies combines the features of each of the components, yielding a decent segmentation of the initial image.

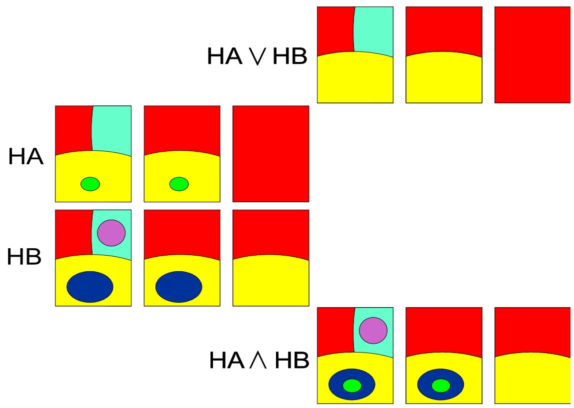

Figure 4: Two hierarchies HA and HB and their derived supremum and infimum

8.3 Lexicographic fusion of stratified hierarchies

Let $\mathcal{A}$ and $\mathcal{B}$ be two stratified hierarchies, with their associated distances $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$. In some cases, one of the hierarchies correctly represents the image to segment, but with a too small number of nested partitions. One desires to enrich the current ranking of regions as given by $\mathcal{A}$, by introducing some intermediate levels in the hierarchy. The solution is to combine the hierarchy $\mathcal{A}$ with another hierarchy $\mathcal{B}$ in a lexicographic order.

One produces the lexicographic hierarchy $\mathrm{Lex}(\mathcal{A}, \mathcal{B})$ by defining its ultrametric distance ; it is the largest ultrametric distance below the lexicographic distance $d_{\mathcal{A},\mathcal{B}}$ classically defined by

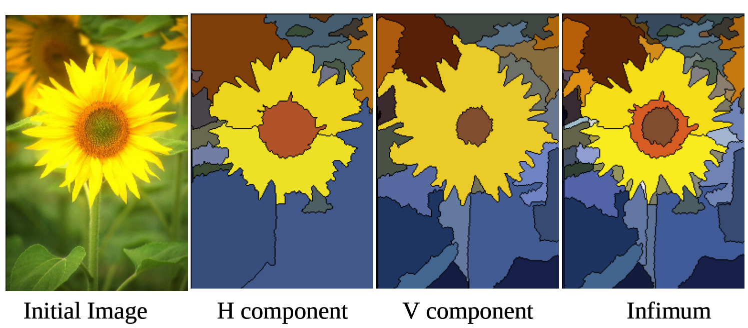

Figure 5: Supremum of two hierarchies.

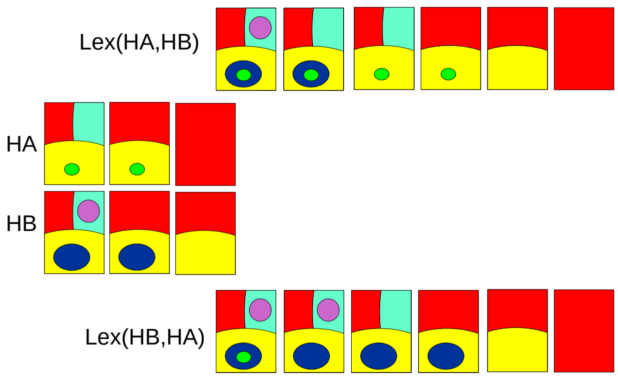

Figure 6: Two hierarchies HA and HB and their derived lexicographic combinations.

$$d_{\mathcal{A},\mathcal{B}}(C,D) > d_{\mathcal{A},\mathcal{B}}(K,L) \Leftrightarrow$$
$$d_{\mathcal{A}}(C,D) > d_{\mathcal{A}}(K,L)$$
$$or$$
$$d_{\mathcal{A}}(C,D) = d_{\mathcal{A}}(K,L) \text{ and } d_{\mathcal{B}}(C,D) > d_{\mathcal{B}}(K,L)$$

Fig.6 present two hierarchies HA and HB and the derived lexicographic hierarchies Lex$(\mathcal{A},\mathcal{B})$ and Lex$(\mathcal{B},\mathcal{A})$. Fig.7 shows an image which is difficult to segment as it contains small contrasted objects, the cars and the landscape and road which are much larger and less contrasted. Two separate segmentation have been performed. The first based on the contrast segments the cars ; the second, based on the "volume" (area of the regions multiplied by the contrast) segments the landscape. The hierarchy of both these segmentations has been thresholded so as to show 30 regions. The lexicographic fusion of both segmentations Lex$(Depth, Volume)$, also thresholded at 30 regions offers a nice composition of both segmentations.

9 Adjunctions on partial hierarchies

We propose two adjunctions, a finer and a coarser adjunction on hierarchies or partial hierarchies. The finer one extends to partial hierarchies the adjunction proposed by J.Serra on partitions [12], extended by Ch. Ronse on partial partitions [7]. The coarser one is presented first. It is obtained by taking the supremum and infimum of PUHDs translated by the translations associated to a structuring element.

Everything presented below is valid for hierarchies and partial hierarchies.

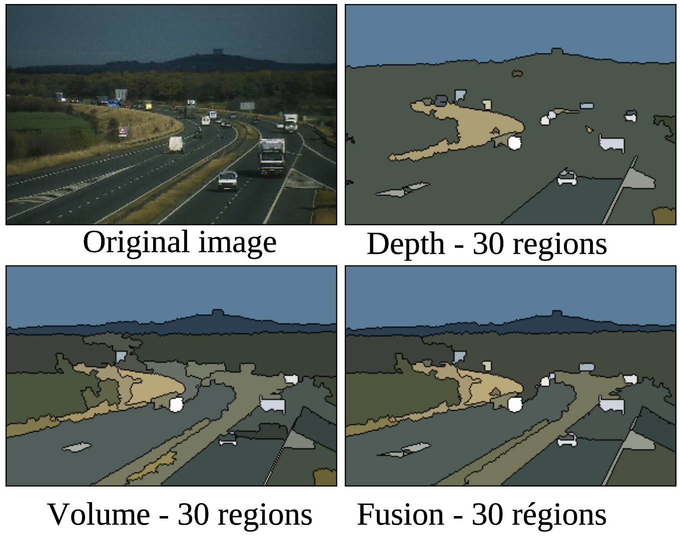

Figure 7: Lexicographic fusion of two hierarchies

9.1 A first adjunction based on the supremum and infimum of translated PUHD

9.1.1 Definition

Given a point O serving as origin, a structuring element B is a family of translations $\bigcup \left\{ \overrightarrow{Ox} \mid x \in B \right\}$. A set X of $\mathcal{P}(E)$ may then be eroded and dilated by this structuring element : the erosion $X \ominus B = \bigwedge_{x \in B} X_{\overrightarrow{Ox}}$ and the dilation $X \oplus B = \bigvee_{x \in B} X_{\overrightarrow{xO}}$. As one uses for one operator the vectors $\overrightarrow{Ox}$ and for the other the vectors $-\overrightarrow{Ox} = \overrightarrow{xO}$, both operators form an adjunction: for any $X, Y \in \mathcal{P}(E)$, we have $X \oplus B < Y \Leftrightarrow X < Y \ominus B$.

A hierarchy $\mathcal{X} \in \mathcal{X}(E)$ is a collection of sets $X^i \in \mathcal{P}(E)$. Through the translation by a vector $\overrightarrow{t}$, these sets $X^i_{\overrightarrow{t}}$ form a new hierarcy $\mathcal{X}_{\overrightarrow{t}}$. If χ is the ultrametric ecart associated to $\mathcal{X}$, the ultrametric ecart associated to $\mathcal{X}_{\overrightarrow{t}}$ will be written $\chi_{\overrightarrow{t}}$.

As the partial hierarchies form a complete lattice $\mathcal{X}(E)$, we may use the same mechanism for constructing an erosion and a dilation on hierarchies. We define two operators operating on a hierarchy $\mathcal{X}$. For showing that the first $\mathcal{X} \ominus B = \bigwedge_{x \in B} \mathcal{X}_{\overrightarrow{Ox}}$ is an erosion and the second $\mathcal{X} \oplus B = \bigvee_{x \in B} \mathcal{X}_{\overrightarrow{xO}}$ a dilation, we have to show that they form an adjunction.

9.1.2 Proof of the adjunction

We have to prove that for any two hierarchies $\mathcal{X}, \mathcal{Y} \in \mathcal{X}(E) : \mathcal{X} \oplus B < \mathcal{Y} \Leftrightarrow \mathcal{X} < \mathcal{Y} \ominus B$.

We will prove the adjunction through the half distance associated to the hierarchies $\mathcal{X}$ and $\mathcal{Y}$.

We have the following correspondances between the hierarchies and the ultrametric ecarts :

- $\mathcal{X} \leftrightarrow \chi$
- $\mathcal{Y} \leftrightarrow \zeta$
- $\mathcal{Y} \ominus B = \bigwedge\limits_{x\in B} \mathcal{Y}_{\overrightarrow{Ox}} \leftrightarrow \bigvee\limits_{x\in B} \zeta_{\overrightarrow{Ox}}$
- $\mathcal{X} \oplus B = \bigvee\limits_{x\in B} \mathcal{X}_{\overrightarrow{xO}} \leftrightarrow \overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}}$
- $\mathcal{X} \oplus B < \mathcal{Y} \Leftrightarrow \mathcal{X} < \mathcal{Y} \ominus B \leftrightarrow \overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}} > \zeta \Leftrightarrow \chi > \bigvee\limits_{x\in B} \zeta_{\overrightarrow{Ox}}$

Let us now prove this last adjunction.
For two arbitrary ultrametric ecarts χ and ζ : $\mathcal{X} < \mathcal{Y} \ominus B \Leftrightarrow \chi > \bigvee\limits_{x\in B} \zeta_{\overrightarrow{Ox}} \Leftrightarrow \forall x \in B : \chi > \zeta_{\overrightarrow{Ox}} \Leftrightarrow \forall x \in B : \chi_{\overrightarrow{xO}} > \zeta \Leftrightarrow \bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}} > \zeta$

Remains to establish : $\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}} > \zeta \Leftrightarrow \overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}} > \zeta$:

- $\overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}} > \zeta \Rightarrow \bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}} > \zeta$ since $\overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}}$ is the largest ultrametric ecart below $\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}$
- Suppose now $\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}} > \zeta$. Since ζ is an ultrametric ecart below $\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}$, it is smaller or equal to the largest ultrametric ecart below $\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}$, that is $\overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}}$

This completes the proof :

$\mathcal{X} < \mathcal{Y} \ominus B \Leftrightarrow \chi > \bigvee\limits_{x\in B} \zeta_{\overrightarrow{Ox}} \Leftrightarrow \bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}} > \zeta \Leftrightarrow \overbrace{\bigwedge\limits_{x\in B} \chi_{\overrightarrow{xO}}} > \zeta \Leftrightarrow \mathcal{X} \oplus B < \mathcal{Y}$

The erosion of a partition by a square structuring element (8 connexity) is illustrated in fig.8, where the smallest squares represent each a pixel.

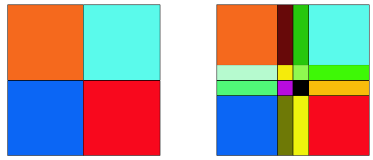

Figure 8: Erosion of a partition by a structuring element equal to the central point and its ' nearest neighbors. The smallest dots in the right picture show the size of the individual pixels in a square raster. Two neighboring pixels p and q belong to the same region of the eroded partition if there exists a $b \in B$ such that $p+b$ and $q+b$ both belong to the same tile of the initial partition.

9.1.3 Expression of the erosion and dilation, valid for hierarchies and partial hierarchies

Consider now a partial herarchy $\mathcal{X}$. We have the following correspondances between the hierarchies and the ultrametric ecarts :

- $\mathcal{X} \leftrightarrow \chi$
- $\mathcal{X} \ominus B = \bigwedge\limits_{x \in B} \mathcal{X}_{\overrightarrow{Ox}} \leftrightarrow \bigvee\limits_{b \in B} \chi_{\overrightarrow{Ox}}$
- $\mathcal{X} \oplus B = \bigvee\limits_{x \in B} \mathcal{X}_{\overrightarrow{xO}} \leftrightarrow \overbrace{\bigwedge\limits_{b \in B} \chi_{\overrightarrow{xO}}}$

The expression of the PUHD is

$\mathcal{X} \ominus B(p,q) = \left[\bigvee\limits_{b \in B} \chi_{\overrightarrow{Ox}}\right](p,q) = \bigvee \{\chi(p+b, q+b) \mid b \in B\}$

$\mathcal{X} \oplus B(p,q) = \left[\overbrace{\bigwedge\limits_{b \in B} \chi_{\overrightarrow{xO}}}\right](p,q) = \overbrace{\bigwedge \{\chi(p-b, q-b) \mid b \in B\}}$

If $p \oplus B$ contains an alien $x = p+b$ with a stratification level λ, then $\chi(p+b, q+b) \geq \lambda$ and $\mathcal{X} \ominus B(p,q) \geq \lambda$.

9.1.4 Illustration

We illustrate the erosion and the opening of a one dimensional hierarchy, first by a structuring element reduced to two pixels, then by a structuring element made of three pixels. In the first case, the erosion and the dilation have to use the structuring element for the erosion and its transposed version for the dilation.

Erosion and opening by a pair of 2 pixels.

Erosion and opening by a centered segment of 3 pixels.

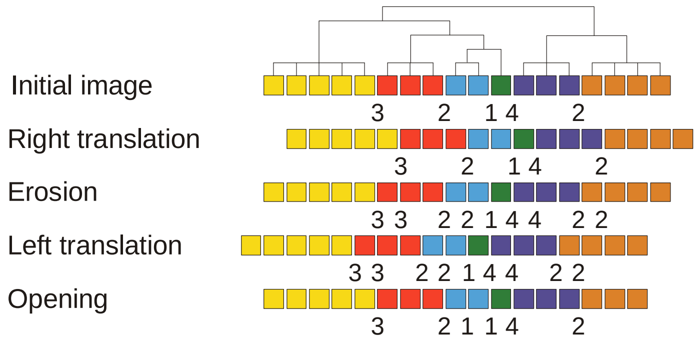

Figure 9: Erosion and opening by a pair of pixels: intermediate steps

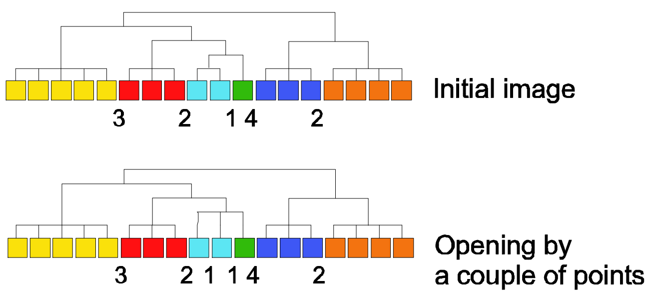

Figure 10: Dendrogram of an initial image and its opening by a segment of 2 points.

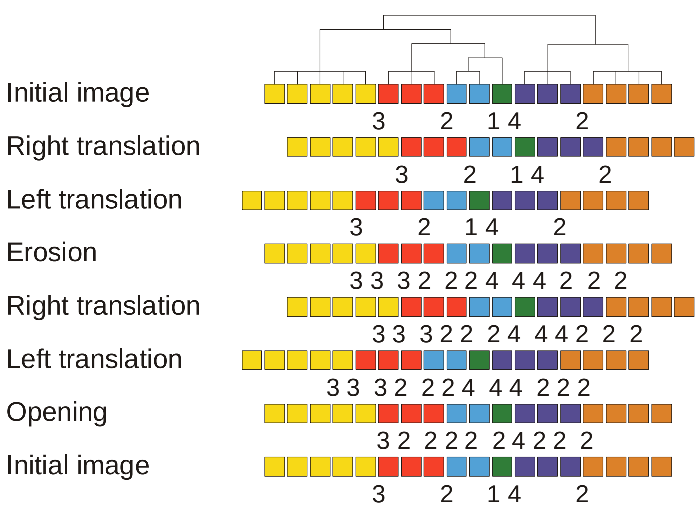

Figure 11: Erosion and opening by a segment of 3 pixels: intermediate steps

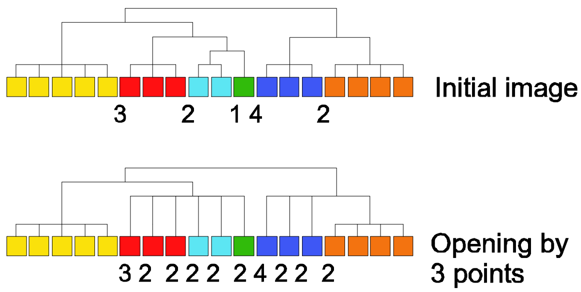

Figure 12: Dendrogram of an initial image and its opening by a segment of 3 points.

9.2 Adjunction on hierarchies and partial hierarchies, defined on a tile by tile basis

In this section we recall how J.Serra defined an adjunction on partitions and how Ch. Ronse adapted it to partial partitions. We illustrate the method and compare with the adjunction presented previously, based on the supremum and infimum of translated partitions. We then extend to partial hierarchies the adjunction defined by Ch. Ronse for partial partition.

9.2.1 Dilation/erosion on partitions

Description of the algorithm J.Serra proposed in [12] where each tile of the partition is eroded separately. As the resulting collection of sets does not cover the domain E, he completes the empty spaces with singletons.

The adjunct dilation has been defined by Ch. Ronse. Il consists in dilating all non singleton sets of a partition, chain all dilated sets with a non empty intersection ; if there are empty spaces, complete with singletons in order to obtain a partition.

Discussion The proposed erosion does not make the distinction between singletons produced by the erosion of some tile of the initial partition and singletons added to fill empty spaces. For this reason, the adjunct dilation dilates only the non singleton parts of a partition and chains all dilated sets with a non empty intersection. If there are spaces left empty, they are filled by singletons.

An opening and a closing can be classically obtained by chaining the erosion and dilation of the preceding adjunction. The singletons produced by a first erosion are discarded by the subsequent opening. Like that, a tile which is identical with the structuring element is pulverized into singletons by the opening.

It is to note that the singletons form a role apart from any other set, as the chaining between sets with non empty intersection can never pass through singletons : the intersection of a set and a singleton is always reduced to the singleton itself.

9.2.2 Adjunction on partial partitions

Description of the algorithm Using partial partitions alleviates this difficulty as shown by Ch. Ronse. The erosion of a partial partition consists in eroding each tile of the partial partition separately, producing a new partial partition, whose support contains all eroded sets produced, including the singletons. Therefore there is no need to complete the empty spaces with singletons, as the support of the partial partitions varies.

The dilation consists in dilating all tiles of the partial partition (including the singletons) and chain all dilated sets with a non empty intersection. The support of the initial partition may like that also be dilated, in order to contain all sets produced by the dilation. Here again, there is no need to fill empty spaces with singletons.

Discussion Here, there is no need of filling singletons, as the support of the partial partition is variable. If after an erosion, there exist singletons in the resulting partial partitions, they duly correspond to eroded sets of the initial partial partition. Therefore they may be dilated to obtain the openings.

9.2.3 Adjunctions on hierarchies and partial hierarchies

In this section, we establish the PUHD (partial ultrametric half distance) for Ronse's adjunction for partial partitions. It happens that the obtained expression is also valid for hierarchies and partial hierarchies.

Let χ be the PUHD (partial ultrametric half distance) representing a partial hierarchy and $(\varepsilon\chi, \delta\chi)$ the adjunction of the PUHDs.

Adjunctions on partitions and partial partitions

Erosion We illustrate the method with a partition, as illustrated in fig.13. The points p and q belong to the same tile of the partition eroded by a structuring element B, if they are centers of disks entirely included in the same tile of the initial partition. In such a case all pairs $x, y \in B_p \cup B_q$ belong to the same tile of the partition, hence $\chi(x, y) = 0$. For the pair p, q we have $\varepsilon\chi(p, q) = 0$. Inversely if these conditions are not verified, there exists a pair of pixels $x, y \in B_p \cup B_q$ which does not belong to the same tile of the partition χ and $\chi(x, y) = 1$. It follows from this analysis, that the PUHD $\varepsilon\chi$ of the eroded hierarchy can be expressed as

$$\varepsilon\chi(p, q) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_q\}$$

Consider now a point p such that B_p is not included in any tile of the partition χ. If $s \in B_p$, there exists then a $t \in B_p$ such that $\chi(s, t) = 1$, otherwise B_p would belong to the same tile of the partition. For such a point p we have $\varepsilon\chi(p, p) = \bigvee \{\chi(x, y) \mid x, y \in B_p \cup B_p\} = 1$, showing that it is an

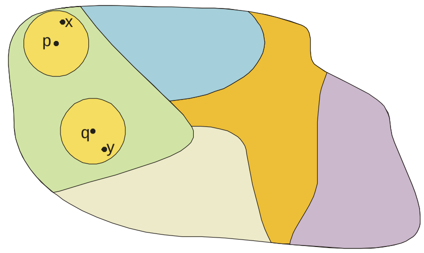

Figure 13: The points p and q belong to the same tile of the partition eroded by a disk, as they are centers of disks entirely included in the same tile of the initial partition.

alien in the eroded partition χ. On the other hand if there exists a tile of the partition containing B_p and the erosion of this tile is reduced to a singleton p, then $\varepsilon\chi(p,p) = \bigvee \{\chi(x,y) \mid x, y \in B_p \cup B_p\} = 0$.

This shows that our formulation faithfully represents the proposition of Ch. Ronse for the erosion in partial partitions.

Dilation We illustrate the method with a partition, as illustrated in fig.14. The points x and y belong to the same tile of the partition dilated by a structuring element B, if there exist two nodes p and q belonging to the same tile of this partition and $x, y \in B_p \cup B_p$. In such a case $\chi(p,q) = 0$. As we look for all pairs p and q verifying these conditions, we have to consider $\bigwedge \{\chi(p,q) \mid x, y \in B_p \cup B_q\}$.

But as soon as there exists a node x belonging simultaneously to B_p and B_q for two nodes p and q belonging to distinct tiles of the initial partial partition, then $B_p \cup B_q$ also belongs to a unique tile of the dilated partial partition. This is the classical situation where we have an infimum of PUHD and we have to consider the corresponding subdominant ultrametric distance to represent the final result. We thus get for the dilation

$$\delta\chi(x,y) = \overbrace{\bigwedge \{\chi(p,q) \mid x, y \in B_p \cup B_q\}}$$

If a set B_p only contains singletons, then $\bigwedge \{\chi(p,q) \mid x, y \in B_p\} = 1$. If on the contrary p is a singleton, then $\bigwedge \{\chi(p,q) \mid x, y \in B_p\} = 0$, showing that singletons get dilated. This again is conform to the description of dilations given by Ch. Ronse.

Adjunctions on hierarchies and partial hierarchies As a matter of fact both expressions established for partitions and partial partitions are still valid for hierarchies and partial hierarchies. If $\mathcal{X}$ is a partial hierarchy and χ its associated PUHD, then we have the classical correspondances :

- to the erosion $\varepsilon\mathcal{X}$ corresponds the dilation of its PUHD $\delta\chi$

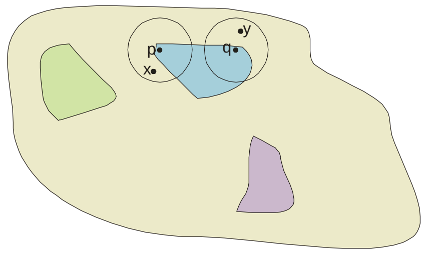

Figure 14: The points x and y belong to the same tile of the dilated partial partition, if there exist two nodes p and q such that B_p and B_q belong to the same tile of this partition.

- to the dilation $\delta\mathcal{X}$ corresponds the erosion of its PUHD $\varepsilon\chi$

The PUHD $\delta\chi$ of the eroded hierarchy $\varepsilon\mathcal{X}$ can be expressed as

$$\varepsilon\mathcal{X} \leftrightarrow \delta\chi(p,q) = \bigvee \{\chi(x,y) \mid x,y \in B_p \cup B_q\}$$

If a set B_p contains an alien x, for which $\chi(x,y) = \lambda$, then $\varepsilon\chi(p,q) = \bigvee \{\chi(x,y) \mid x,y \in B_p \cup B_q\} \geq \chi(x,y) = \lambda$, showing that the pixel x can only appear in the erosion of the hierarchy at a stratification level higher than or equal to λ.

The support $\mathring{\mathrm{supp}}_\lambda \varepsilon\mathcal{X} = \mathring{\mathrm{supp}}_\lambda \delta\chi$ A point p belongs to $\mathring{\mathrm{supp}}_\lambda(\delta\mathcal{X})$ as soon as $\bigvee \{\chi(x,y) \mid x,y \in B_p\} < \lambda$ which is equivalent with $\forall x,y \in B_p : \chi(x,y) < \lambda$, which implies $\forall x \in B_p : \chi(x,x) < \lambda$, showing that p belongs to $\mathring{\mathrm{supp}}_\lambda(\mathcal{X}) \ominus B$. Hence $\mathring{\mathrm{supp}}_\lambda(\delta\mathcal{X}) \subset \mathring{\mathrm{supp}}_\lambda(\mathcal{X}) \ominus B$

Dilation We thus get for the dilation :

$$\delta\mathcal{X} \leftrightarrow \varepsilon\chi(x,y) = \overbrace{\bigwedge \{\chi(p,q) \mid x,y \in B_p \cup B_q\}}$$

If a set B_p only contains singletons, then $\bigwedge \{\chi(p,q) \mid x,y \in B_p\} = 1$. If on the contrary p is a singleton, then $\bigwedge \{\chi(p,q) \mid x,y \in B_p\} = 0$, showing that singletons get dilated. This again is conform to the description of dilations given by Ch. Ronse.

The support $\overline{\mathrm{supp}}_\lambda \delta\mathcal{X} = \overline{\mathrm{supp}}_\lambda \varepsilon\chi$ A point p belongs to $\overline{\mathrm{supp}}_\lambda(\varepsilon\chi)$ as soon as $\varepsilon\chi(p,p) = \lambda$. A sufficient condition is the existence of a $b \in B$ such that $\chi(p-b,p-b) = \lambda$; this will be the case if p belongs to $\overline{\mathrm{supp}}_\lambda(\chi)$ dilated by B. Hence $\overline{\mathrm{supp}}_\lambda(\chi) \subset \overline{\mathrm{supp}}_\lambda(\varepsilon\chi)$. Inversely if $p \notin \overline{\mathrm{supp}}_\lambda(\chi) \oplus B$, then there exists a $b \in B$, such that $\chi(p-b,p-b) > \lambda$. But then for any q we have $\chi(p-b,q) > \lambda$, showing that $\bigwedge \{\chi(p-b_1,q-b_2) \mid b_1,b_2 \in B\} > \lambda$. In this case, the sup-section of any path between p and q is necessarily higher or equal to λ and $p \notin \overline{\text{supp}}_\lambda(\varepsilon\chi)$. Finally we have shown that $\overline{\text{supp}}_\lambda(\varepsilon\chi) = \overline{\text{supp}}_\lambda(\delta\mathcal{X}) = \overline{\text{supp}}_\lambda(\chi) \oplus B$.

We will give below an equivalent formulation of these operators and show that they indeed form an adjunction.

9.3 Comparison of the three adjunction for partitions and partial partitions.

We have now three distinct adjunctions on partitions. We compare them in fig.15 by first eroding a partition and then dilating the result with the adjunct dilation. The structuring element is a cross, made of the central pixel and its four nearest neighbors. Let Π be a partition.

9.3.1 Partitions : Tile by tile construction, filling with singletons (Serra, Ronse)

Erosion The adjunction on partitions, proposed by J. Serra may be expressed as follows. Two points p and q belong to the same tile of the eroded partition $\varepsilon\Pi$, by a structuring element B if and only if there exists a tile A of the initial partition Π, such that for each b belonging to B, $p+b$ and $q+b$ belong to A. This condition may be reformulated as follows.

Proposition 22 *Two points p and q belong to the same tile of the eroded partition $\varepsilon\Pi$, by a structuring element B if and only if for each $b_1 \in B$ and each $b_2 \in B$, $p+b_1$ and $q+b_2$belong to the same tile of the initial partition.*

Proof.
a) Suppose that there exists a tile A of the initial partition Π such that for each b belonging to B, $p+b$ and $q+b$ belong to A. Then $p \oplus B$ and $q \oplus B$ belong to A and $\forall b_1, b_2 \in B, p+b_1$ and $q+b_2$ belong to A
b) Inversely suppose that for each $b_1 \in B$ and each $b_2 \in B$, $p+b_1$ and $q+b_2$ belong to the same tile of the initial partition. Fixing $b_1 = u$ and varying b_2 shows that $q \oplus B$ belongs to a same tile A as $p+u$; then fixing $b_2 = v$ and varying b_1 shows that $p \oplus B$ also belongs to this tile A. Hence there exists a tile A of the initial partition Π such that for each b belonging to B, $p+b$ and $q+b$ belong to A. ■

We derive from this proposition a new and equivalent expression for the erosion :

$$\varepsilon\mathcal{X}(p,q) \leftrightarrow \delta\chi(p,q) = \bigvee \{\chi(p+b_1, q+b_2) \mid b_1, b_2 \in B\}$$

Dilation Two points x and y belong to the same tile of the dilated partition $\delta\Pi$ by a structuring element B only if there exists a non singleton tile A of the initial partition and two points p and q in A, such that x and y belong to the dilation of these points $p \oplus B$ and $q \oplus B$; this will be the case if and only if there exist b_1 and b_2 belonging to B such that $x = p + b_1$ and $y = q + b_2$. This condition may be reformulated as follows.

Proposition 23 *Two points x and y belong to the same tile of the dilated partition $\delta\Pi$ by a structuring element B if and only if there exist b_1 and b_2 belonging to B such that $x - b_1$ and $y - b_2$ both belong to the same tile of the initial partition.*

After transitive closure, one gets :

$$\delta\mathcal{X}(x, y) \leftrightarrow \varepsilon\chi(x, y) = \overbrace{\bigwedge \{\chi(x - b_1, y - b_2) \mid b_1, b_2 \in B\}}$$

Adjunction The couple $(\varepsilon\mathcal{X}, \delta\mathcal{X})$ forms an adjunction for the partial hierarchies. In order to prove it, we show that $(\varepsilon\chi, \delta\chi)$ forms an adjunction for the PUHDs.

Proof.

We have to show that for any couple of PUHD $\chi, \zeta : \delta\chi \leq \zeta \Leftrightarrow \chi \leq \varepsilon\zeta$

$\forall p, q \in E : \bigvee \{\chi(p + b_1, q + b_2) \mid b_1, b_2 \in B\} \leq \zeta(p, q) \Leftrightarrow \forall b_1, b_2 \in B : \chi(p + b_1, q + b_2) \leq \zeta(p, q) \Leftrightarrow$

$\forall b_1, b_2 \in B : \chi(p, q) \leq \zeta(p{-}b1, q{-}b2) \Leftrightarrow \chi(p, q) \leq \bigwedge \{\zeta(p - b_1, q - b_2) \mid b_1, b_2 \in B\}$

And $\chi(p, q)$, being a PUHD smaller than $\{\zeta(p - b_1, q - b_2) \mid b_1, b_2 \in B\}$ is smaller than the subdominant ultrametric smaller than $\{\zeta(p - b_1, q - b_2) \mid b_1, b_2 \in B\}$,

i.e. $\chi(p, q) \leq \overbrace{\bigwedge \{\chi(p - b_1, q - b_2) \mid b_1, b_2 \in B\}}$.

Inversely if $\chi(p, q) \leq \overbrace{\bigwedge \{\chi(p - b_1, q - b_2) \mid b_1, b_2 \in B\}}$,

then $\chi(p, q) \leq \bigwedge \{\chi(p - b_1, q - b_2) \mid b_1, b_2 \in B\}$

since $\overbrace{\bigwedge \{\chi(p - b_1, q - b_2) \mid b_1, b_2 \in B\}} \leq \bigwedge \{\chi(p - b_1, q - b_2) \mid b_1, b_2 \in B\}$ ■

Illustration This couple erosion/opening is presented in the first column of two images under the heading "Tile by tile, filling with singletons" in fig.15. The singletons which are produced get a uniform yellow colour.

Fig.16 shows the behaviour of 3 pixels. After the erosion, the empty spaces are replaced by singletons, are they are not centers of a structuring element included in a tile of the initial partition. On the contrary, the central square is eroded into a singleton. There is no means to distinguish this singleton from the "filling singletons" at positions p and q. During the subsequent dilation, p and q belong to the dilation of the pixels on their right, belonging to the green tile. The pixel x on the contrary does not belong to any tile of the erosion and remains a singleton in the opening.

9.3.2 Partial partitions : Tile by tile construction, adjusting the support (Ronse)

The adjunction on partial partitions, identical with the preceding, except that the support of the erosion and dilations varies, avoiding the need to fill the empty spaces with singletons. If π is the PUHD of a partial partition Π, the PUHD of their erosion and dilation esstablished just above is still valid.

Two points p and q belong to the same tile of the eroded partition $\varepsilon\Pi$, by a structuring element B if and only if for each $b_1 \in B$ and each $b_2 \in B$, $p+b_1$ and $q+b_2$belong to the same tile of the initial partition. We derive a new expression for the erosion :

$$\varepsilon\mathcal{X}(p,q) \leftrightarrow \delta\chi(p,q) = \bigvee \{\chi(p+b_1, q+b_2) \mid b_1, b_2 \in B\}$$

Two points p and q belong to the same tile of the dilated partition $\delta\Pi$ if and only if there exist b_1 and b_2 belonging to B such that $p-b_1$ and $q-b_2$ belong to the same tile of the initial partition. After chaining the regions with a non empty intersection, one gets :

$$\delta\mathcal{X}(x,y) \leftrightarrow \varepsilon\chi(x,y) = \overbrace{\bigwedge \{\chi(p-b_1, q-b_2) \mid b_1, b_2 \in B\}}$$

This couple erosion/opening is presented in the first column of two images under the heading "Tile by tile on partial partitions, adjusting the support" in fig.15. Here singletons produced by the erosion remain singletons and constitute a valid tile of the eroded partial partition. This is the case for the green singleton, result of eroding the green square in the original partition. All empty spaces are expelled out of the support of the partial partition and are represented here in grey. The subsequent dilation dilates only the pixels inside the support of the erosion. Like that the green singleton left in the erosion is dilated, extending again the support of the erosion. The pixels left outside the support of the opening are represented in grey.
Fig.16 shows the behaviour of 3 pixels. During the erosion, the pixels p, q and x are expelled from the support of the partial eroded partition, are they are not centers of a structuring element included in a tile of the initial partition. During the subsequent dilation, p and q belong to the dilation of the pixels on their right, belonging to the green tile. The pixel x on the contrary does not belong to any tile of the erosion and remains outside the support of the opening.

9.3.3 Partial hierarchies: supremum and infimum of PUHD

The adjunction based on the supremum and infimum of translated hierarchies is illustrated in fig.15 under the heading "By supremum and infimum of ultrametric 1/2 distances".
Two points p and q belong to the same tile of the eroded partition $\varepsilon\Pi$ by a structuring element B if and only if, for for each b belonging to B there exists a tile A of the initial partition Π, such that $p+b$ and $q+b$ belong to A. This is the case for the pixels p and q in fig.16. The pixel x on the contrary becomes a singleton.
Two points x and y belong to the same tile of the dilated partition $\delta\Pi$ by a structuring element B only if there exists a tile A of the initial partition, two points p and q in A and a b belonging to B, such that $x = p+b$ and $y = q+b$. Consider again the pixels p and q in fig.16. They belong to the same tile of the

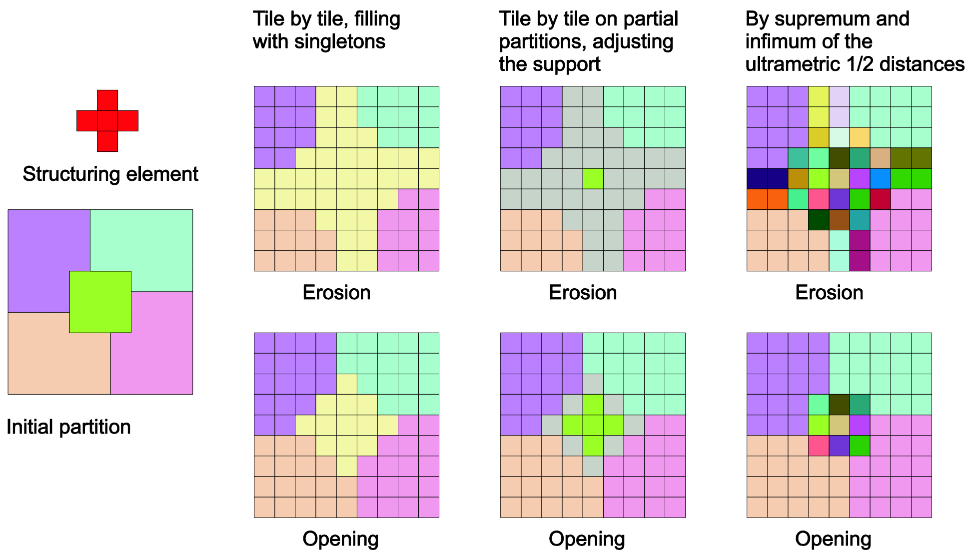

Figure 15: Erosion and opening of a partition, obtained by chaining erosion and dilation. The first couple operates tile by tile with completion with singletons (Serra), the second adjusts the domain and works on partial partitions (Ronse), the last is based on the supremum and infimum of ultrametric half distances (Meyer)

dilated partition, as there right neighbors belong to the same tile of the eroded partition. Pixels q and x also belong to a same tile as their upper neighbors belong to the same tile of the eroded partition. Finally, through chaining at q, all three pixels belong to the same tile of the opening.

9.3.4 Ordering the adjunctions on partial hierarchies or partitions

Consider a partial partition $\mathcal{X}$. We have the following correspondances between the hierarchies and the ultrametric ecarts : $\mathcal{X} \leftrightarrow \chi$. We just obtained the expression of the PUHD for the adjunction tile par tile $(\varepsilon\mathcal{X}, \delta\mathcal{X})$.

- $\varepsilon\mathcal{X} =$

We earlier obtained the expression of the PUHD for the adjunction $(\mathcal{X} \ominus B, \mathcal{X} \oplus B) = \left(\bigwedge\limits_{x \in B} \mathcal{X}_{\overrightarrow{Ox}}, \bigvee\limits_{x \in B} \mathcal{X}_{\overrightarrow{xO}} \right)$:

We have the following correspondances between the hierarchies and the corresponding PUHDs:

- $\mathcal{X} \ominus B = \bigwedge\limits_{x \in B} \mathcal{X}_{\overrightarrow{Ox}} \leftrightarrow \bigvee\limits_{b \in B} \chi_{\overrightarrow{Ox}} = \chi \oplus B$

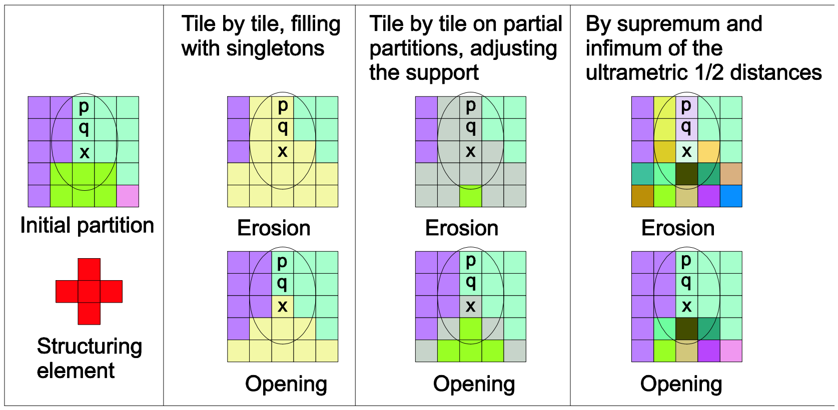

Figure 16: Zoom on 3 pixels. Erosion and opening of a partition, obtained by chaining erosion and dilation. The first couple operates tile by tile with completion with singletons (Serra), the second adjusts the domain and works on partial partitions (Ronse), the last is based on the supremum and infimum of ultrametric half distances (Meyer)

- $\mathcal{X} \oplus B = \bigvee\limits_{x\in B} \mathcal{X}_{\overrightarrow{xO}} \leftrightarrow \overbrace{\bigwedge\limits_{b\in B} \chi_{\overrightarrow{xO}}} = \chi \ominus B$

The expression of the PUHD is

- $\chi \oplus B(p,q) = \bigvee \{\chi(p+b, q+b) \mid b \in B\}$
- $\chi \ominus B(p,q) = \overbrace{\bigwedge \{\chi(p-b, q-b) \mid b \in B\}}$

These expressions may be compared with the tile by tile erosion and dilations, which are expressed by

- $\varepsilon\mathcal{X}(p,q) \leftrightarrow \delta\chi(p,q) = \bigvee \{\chi(x,y) \mid x, y \in B_p \cup B_q\} = \bigvee \{\chi(p+b_1, q+b_2) \mid b_1, b_2 \in B\}$
- $\delta\mathcal{X}(p,q) \leftrightarrow \varepsilon\chi(p,q) = \overbrace{\bigwedge \{\chi(x,y) \mid p, q \in B_x \cup B_y\}} = \overbrace{\bigwedge \{\chi(p-b_1, q-b_2) \mid b_1, b_2 \in B\}}$

These expression verify the following order relations :

- $\bigvee \{\chi(p+b, q+b) \mid b \in B\} \ \leq \ \bigvee \{\chi(p+b_1, q+b_2) \mid b_1, b_2 \in B\}$, showing that the partial hierarchy $\varepsilon\mathcal{X}$ is coarser than the partial hierarchy $\mathcal{X} \ominus B$
- $\overbrace{\bigwedge \{\chi(p-b_1, q-b_2) \mid b_1, b_2 \in B\}} \leq \overbrace{\bigwedge \{\chi(p-b, q-b) \mid b \in B\}}$ showing that the partial hierarchy $\delta\mathcal{X}$ is finer than the partial hierarchy $\mathcal{X} \oplus B$

If the origin belongs to the structuring element we have the following order relations between the partial hierarchies :

$$\varepsilon\mathcal{X} \leq \mathcal{X} \ominus B \leq \mathcal{X} \leq \mathcal{X} \oplus B \leq \delta\mathcal{X}$$

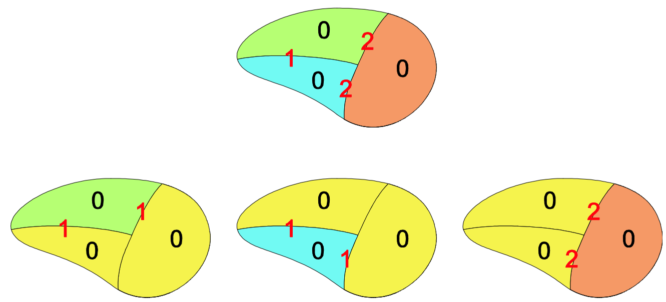

Figure 17: Min-decomposition of a hierarchy into three elementary hierarchies.

9.4 Decomposition and recomposition of hierarchies

The next section proposes an inf-decomposition and a sup decomposition of hierarchies, which are useful for analysing the behaviour of the various adjunctions working on hierarchies, as erosion commute with infimum and dilation with supremum.

Let $\mathcal{X}$ be a partial hierarchy with its associated PUHD χ.

9.4.1 Inf-generation of a partial hierarchy i.e sup-generation of its PUHD

For each pair p, q we want to construct a 2 regions hierarchy χ_{pq} where the distance $\chi(p,q)$ is well represented and all other couple of points r, s take a value which is inferior :$\chi_{pq}(r,s) \leq \chi(r,s)$.

For $p, q \in E$, one defines $A_{pq} = \overset{\circ}{B}(p, \chi(p,q))$, , the open ball centered at p and q with a radius $\chi(p,q)$. The hierarchy χ_{pq} has only two regions A_{pq} and its complement $\overline{A_{pq}}$. It is defined by:

$$\chi_{pq}(x,y) = \begin{array}{l} 0 \text{ if } x, y \in A_{pq} \text{ or } x, y \in \overline{A_{pq}} \\ \chi(p,q) \text{ otherwise} \end{array}$$

The inf-generation of $\mathcal{X}$ is associated to the PUHD : $\chi = \bigvee\limits_{p,q} \chi_{pq}$

Fig. 17 presents the decomposition into three component of a hierarchy with three leaves. The first hierarchy is obtained for p belonging to the green region and q to the blue one. The second is obtained by exchanging p and q. The last one is obtained for p belonging to the orange region, q being outside. The values of the boundaries of each partial hierarchy are indicated in red.

Analysis of the erosion of partial hierarchies, i.e. dilation of their PUHDs

Case of $\mathcal{X} \ominus B(p,q) \leftrightarrow \left[\bigvee_{b\in B} \chi_{\overrightarrow{Ox}}\right](p,q) = \bigvee\{\chi(p+b,q+b) \mid b \in B\}$ $\chi \oplus B = \left(\bigvee_{p,q} \chi_{pq}\right) \oplus B = \bigvee_{p,q} \chi_{pq} \oplus B$

For $s,t \in E : \chi_{pq} \oplus B(s,t) = \begin{array}{l} 0 \text{ if } \forall b \in B : s+b, t+b \in A \text{ or } s+b, t+b \in \overline{A} \\ \chi(p,q) \text{ if } \exists b \in B : s+b \in A \text{ and } t+b \in \overline{A} \end{array}$

This analysis shows that provided each pair $s+b, t+b$ belongs to the same tile A or $\overline{A}$, then $\chi_{pq} \oplus B(s,t) = 0$

Case of $\varepsilon\mathcal{X}(p,q) \leftrightarrow \delta\chi(p,q) = \bigvee\{\chi(x,y) \mid x,y \in B_p \cup B_q\}$ $= \bigvee\{\chi(p+b_1,q+b_2) \mid b_1,b_2 \in B\}$ $\delta\chi = \delta\left(\bigvee_{p,q} \chi_{pq}\right) = \bigvee_{p,q} \delta\chi_{pq}$

For $s,t \in E : \delta\chi_{pq}(s,t) = \begin{array}{l} 0 \text{ if } \forall b_1,b_2 \in B : s+b_1, t+b_2 \in A \text{ or } s+b_1, t+b_2 \in \overline{A} \\ \chi(p,q) \text{ if } \exists b_1,b_2 \in B : s+b_1 \in A \text{ and } t+b_2 \in \overline{A} \end{array}$

This analysis shows that as soon one point $s+b_1$ belongs to A (resp. $\overline{A}$), the whole element B has to belong to A (resp. $\overline{A}$) to get a value 0 for the dilation of the PUHD.

9.4.2 Sup-generation of a partial hierarchy i.e inf-generation of its PUHD

For each pair p,q we want to construct a 2 regions hierarchy χ^{pq} where the distance $\chi(p,q)$ is well represented and all other couple of points r,s take a value which is superior :$\chi^{pq}(r,s) \geq \chi(r,s)$.

For $p,q \in E$, one defines $A^{pq} = \overline{B}(p,\chi(p,q))$, the closed ball centered at p and q with a radius $\chi(p,q)$. The hierarchy χ^{pq} has only two regions A^{pq} and its complement $\overline{A^{pq}}$. The hierarchy χ^{pq} is defined by

$\chi^{pq}(x,y) = \begin{array}{l} \chi(p,q) \text{ if } x,y \in A^{pq} \\ L \text{ otherwise} \end{array}$

The sup-generation of $\mathcal{X}$ is associated to the PUHD : $\chi = \bigwedge_{p,q} \chi^{pq}$. Hence $\chi = \bigwedge_{p,q} \chi^{pq}$,which is a PUHD and is identical with its transitive closure: $\chi = \bigwedge_{p,q} \chi^{pq} = \overbrace{\bigwedge_{p,q} \chi^{pq}}$.

Fig. 17 presents the decomposition into four component of a hierarchy with three leaves. The values of the boundaries is indicated in red. The first hierarchy is associated to a couple of points, both in the green region. The second to a couple of points, both in the blue region. For the third, one point is in the green region and the other in the blue region. The last is associated to two points, one in the orange region and the other in the green region.

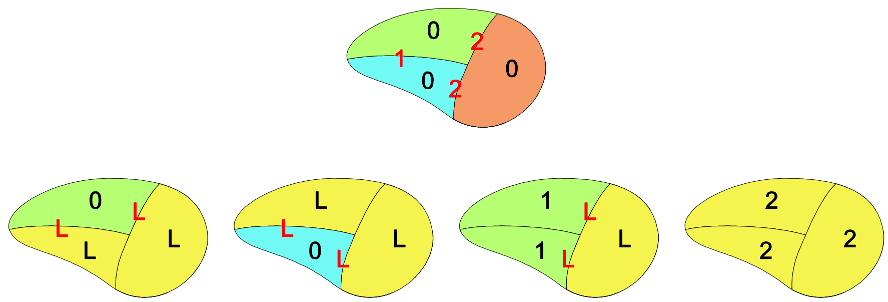

Figure 18: Max-decomposition of a hierarchy into 4 elementary partial hierarchies

9.4.3 Pulse inf-generation of a PUHD

A pulse inf-generation of a PUHD may be defined by considering pairs of pixels, considered as pulses. To each pair $p, q \in E$, one associates a PUHD $\mathbf{1}^{pq}$ taking the value L for all pairs $(s,t) \neq (p,q)$ and the value $\chi(p,q)$ for the pair (p,q).

The PUHD $\mathbf{1}^{pq}$ verifies $\mathbf{1}^{pq} \geq \chi$ with equality for the pair $(p,q) : \mathbf{1}^{pq}(p,q) = \chi(p,q)$.

Hence $\chi = \bigwedge\limits_{p,q} \mathbf{1}^{pq}$,which is a PUHD, hence the transitive closure is not necessary and $\chi = \bigwedge\limits_{p,q} \mathbf{1}^{pq} = \overbrace{\bigwedge\limits_{p,q} \mathbf{1}^{pq}}$.

Analysis of the dilation of partial hierarchies, i.e. erosion of their PUHDs

Case of $\mathcal{X} \oplus B(p,q) \leftrightarrow \chi \ominus B(p,q) = \overbrace{\bigwedge \{\chi(p-b, q-b) \mid b \in B\}}$ $\quad \chi \ominus B = \left(\bigwedge\limits_{p,q} \mathbf{1}^{pq}\right) \ominus B = \overbrace{\bigwedge\limits_{p,q} \mathbf{1}^{pq} \ominus B}$

For $s,t \in E : \mathbf{1}^{pq} \ominus B(s,t) = \begin{array}{l} \chi(p,q) \text{ if } \exists b \in B : p-b = s \text{ and } q-b = t \\ L \text{ otherwise} \end{array}$

This analysis shows that the erosion by B takes a value different of L only for the pairs of points for which there exists a $b \in B$ such that $p - b = s$ and $q - b = t$

Case of $\delta\mathcal{X}(p,q) \leftrightarrow \varepsilon\chi(p,q) = \overbrace{\bigwedge \{\chi(x,y) \mid p,q \in B_x \cup B_y\}}$ $= \overbrace{\bigwedge \{\chi(p-b_1, q-b_2) \mid b_1, b_2 \in B\}}$ $\quad \varepsilon\chi = \varepsilon \left(\bigwedge\limits_{p,q} \mathbf{1}^{pq}\right) = \bigwedge\limits_{p,q} \varepsilon \mathbf{1}^{pq}$

For $s,t \in E : \varepsilon\mathbf{1}^{pq}(s,t) = \begin{array}{l} \chi(p,q) \text{ if } \exists b_1, b_2 \in B : s - b_1 = p \text{ and } t - b_2 = q \\ L \text{ otherwise} \end{array}$

This analysis shows that for all pairs of points (s,t) inside $B_p \cup B_q$, we have $\varepsilon \mathbf{1}^{pq}(s,t) = \chi(p,q)$

10 Some examples of hierarchies

Hierarchies associated to a dissimilarity index A series of nested partitions $(\mathcal{X}_i)$, and hence a hierarchy, may easily be generated from an initial fine partition $\mathcal{X}_0 = \cup R_i,\ i = 1,\dots,n$ on which a dissimilarity index δ is defined between a subset G of all couples of tiles. For a couple of tiles which do not belong to G, we define a dissimilarity equal to ∞.

If we now take the union of all tiles of $\mathcal{X}_0$ with a dissimilarity index below a given threshold λ, we obtain a coarser partition with a stratification index equal to λ. For increasing values of λ we obtain a series of nested partitions, forming a hierarchy $\mathcal{A}$. The ultrametric distance d associated to this hierarchy is precisely the subdominant ultrametric distance associated to δ, that is the largest ultrametric distance below δ (see below the supremum of two hierarchies, where the subdominant ultrametric distance also appears) For two tiles A and B of $\mathcal{X}_0$, the subdominant ultrametric distance will be the lowest level λ for which A and B belong to the same tile (if it does not happen, their distance is ∞)

Other possible measures are color distances, various measures of local contrast, or even motion or texture dissimilarity between adjacent catchment basins.

Case of the watershed tesselation If the tessellation is the result of the watershed construction on a gradient image, the dissimilarity measure can be defined as the altitude of the pass point separating two adjacent regions. The ultrametric half distance between two minima is then the "flooding distance" : the flooding distance between two points p and q is the altitude of the lowest flooding for which p and q both belong to a common lake.

If the flooding is not uniform but increasing with a time parameter τ, then the distance between two points p and q is the time when both points first belong to the same catchment basin of the flooded surface.

The stochastic watershed introduced by J.Angulo [1] is yet another interesting hierarchy, able to produce fine segmentations both on medical image as on multimedia images.

11 Hierarchies for interactive segmentation

11.1 An adjunction associated to a partition Π

To any partition Π on E we may associate a dilation δ. For a point $p \in E$, one defines $\delta(p) = \mathrm{cl}(p)$. One then defines $\delta(X) = \bigcup \{\delta(x) \mid x \in X\} = \bigcup \{C_i \in \Pi \mid X \cap C_i \neq \emptyset\}$

The properties of δ are the following :

- δ is increasing and commutes with union : it is indeed a dilation
- obvioulsy $x \in \delta(x)$, hence δ is extensive
- it is also a closing. The fact that δ also is a closing seems at first sight strange, as the class of invariants of a closing is stable by intersection. But the invariants of δ are unions of classes of the partition Π. Hence their class is stable by intersection. It is easy to check that δ is a dilation-closing : $\delta\delta(Y) = \delta(Y) \Rightarrow \delta(Y) \subset \varepsilon\delta(Y)$ by adjunction ; but ε being anti-extensive, we have $\delta(Y) \subset \varepsilon\delta(Y) \subset \delta(Y)$, hence $\delta(Y) = \varepsilon\delta(Y)$. By duality, we have $\varepsilon = \delta\varepsilon$.

Let us now study the erosion ε adjunct to δ : $Y \subset \varepsilon(X) \Leftrightarrow \delta(Y) \subset X$.

Obviously $\varepsilon(X) = \bigcup \{Y \mid Y \subset \varepsilon(X)\} = \bigcup \{Y \mid \delta(Y) \subset X\}$. But since δ is extensive and idempotent
$\bigcup \{Y \mid \delta(Y) \subset X\} = \bigcup \{\delta(Y) \mid \delta(Y) \subset X\} = \bigcup \{C_i \in \Pi \mid C_i \subset X\}$.

By duality ε is increasing, anti-extensive, idempotent and commutes with intersection, it is an erosion-opening: $\varepsilon = \delta\varepsilon$

11.2 Adjunctions associated to a hierarchy $\mathcal{X}$.

11.2.1 A first adjunction for interactive segmentation

The closed balls $\text{Ball}(x, \rho)$ of radius ρ form a partition, for which we may apply the results of the previous paragraph and define the adjunction $(\delta_\lambda, \varepsilon_\lambda)$ defined by:

- $\delta_\rho(X) = \bigcup \{\text{Ball}(x, \rho) \mid x \in X\}$
- $\varepsilon_\rho(X) = \bigcup \{Y \mid Y \subset \varepsilon(X)\} = \bigcup \{Y \mid \delta(Y) \subset X\}$
  $= \bigcup \{\text{Ball}(x, \rho) \mid x \in E \ , \ \text{Ball}(x, \rho) \subset X\}$

11.2.2 A second adjunction, centered on a point x

The family of balls $\overset{\circ}{\text{Ball}}(x, \mu)$ for increasing values of $\mu \leq \lambda$ is completely ordered for the inclusion. Thus, as recalled by Ch. Ronse in [7], the two following operators form an adjunction :

$\varepsilon_\lambda^x = \bigvee \left\{ \overset{\circ}{\text{Ball}}(x, \mu) : \overset{\circ}{\text{Ball}}(x, \mu) \subset X \right\}$ is an erosion and an opening. It is the largest ball $\overset{\circ}{\text{Ball}}(x, \mu)$ centered in x included in X.

Its adjunct operator $\delta_\lambda^x = \bigwedge \left\{ \overset{\circ}{\text{Ball}}(x, \mu) : X \subset \overset{\circ}{\text{Ball}}(x, \mu) \right\}$ is a dilation and a closing. It is the smallest ball $\overset{\circ}{\text{Ball}}(x, \mu)$ centered in x containing X.

These two operators are useful for interactive segmentation.

11.3 Applications : interactive segmentation

The following examples have been developed within a toolbox for interactive segmentation ([13]).

11.3.1 Intelligent brush

An intelligent brush segments an image by "painting" it: it first selects a zone of interest by painting. Contrary to conventional brushes, the brush adapts its shape to the contours of the image. The shape of the brush is given by the region of the hierarchy containing the cursor. Moving from one place to another changes the shape of the brush, when one goes from one tile of a partition to its neighboring tile. Going up and down the hierarchy modifies the shape of the brush. In fig.19, on the left, one shows the trajectory of the brush ; in the centre, the result of a fixed size brush, and on the right the result of a self-adapting brush following the hierarchy. This self adapting brush is nothing by the dilation δ_r by a ball associated to the hierarchy, centered at the position of the mouse and of a radius, also easily modified through the mouse. This method has been used with success in a package for interactive segmention of organs in 3D medical images.

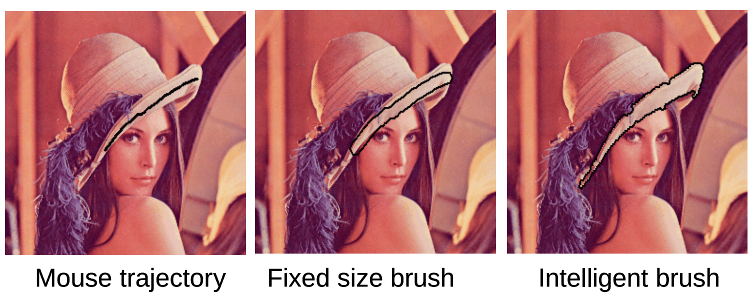

Figure 19: Comparison of the drawing with a fixed size brush and a self adaptive brush.

11.3.2 Magic wand

The magic wand in a conventional computer graphics toolbox consists in extracting the region which touches the position of the mouse and whose colour lies within some predefined limits from the coulour at the mouse position. The next step consists in replacing this set by the smallest set of the hierarchy which contains it. This operation is a dilation and a closing, described by Ch. Ronse as Lemma 8 in [7]. The result is shown in fig.20

Figure 20: Left: initial image
Center: result of the magic wand
Right ; smallest region of the hierarchy containing the magic wand.

12 Connectivity and taxonomy classes

The notion of a connected set in E is well defined if E is a topological space. In [10], Serra generalized this concept by the introduction of a connectivity class. Connectivity classes define the subsets of E which are connected. Hence they help decomposing every set $X \in \mathcal{P}(E)$ into its connected components. Connectivity classes have been extensively studied by Serra and Ronse ([12],[7]) ; these authors later called them connections. A clear presentation of the developments linked to connectivity may be found in [3].

In this last section we show how dendrograms and their associated half distance shed a new light on connections.

12.1 Reminder on connections

Definition 24 *Let E be an arbitrary nonempty set. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a connectivity class or connection if it satisfies*
(C1) $\varnothing \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for $x \in E$
(C2) if $C_i \in \mathcal{C}$ and $\bigcap_{i \in I} C_i \neq \varnothing$, then $\bigcup_{i \in I} C_i \in \mathcal{C}$

Alternatively, we say that $\mathcal{C}$ defines a *connectivity* on E. An element of $\mathcal{C}$ is called a *connected set*. This definition is "generative" : larger connected sets are generated from elementary ones with a non empty intersection.

If we drop the condition $\{x\} \in \mathcal{C}$ for $x \in E$, then we get a partial connection. The union of all sets belonging to $\mathcal{C}$ is called support of $\mathcal{C}$: $\mathrm{supp}(\mathcal{C})$.

12.1.1 Connectivity openings

Serra in [10] has shown that any connected class $\mathcal{C}$ is equivalent with the datum of a connected opening, defined through its invariance domain. If $\mathcal{C}_x$ denotes the subclass of $C \in \mathcal{C}$ that contains a given point ,

$$\mathcal{C}_x = \{C : x \in C \subset \mathcal{C}\}$$

then the union of each non-empty family of sets of $\mathcal{C}_x$, all containing x still belongs to $\mathcal{C}_x$, because of (C2). Hence $\mathrm{Inv}(\gamma_x) = \mathcal{C}_x \cup \{\varnothing\}$ is the invariant set of an opening γ_x, called connected opening of origin x. Its expression is

$$\gamma_x(X) = \bigcup \{C : x \in C \subset \mathcal{C} \text{ and } C \subseteq X\}$$

Since any $x \in E$ belongs to a connected set of $\mathcal{C}$, we have

$$\mathcal{C} = \bigcup_{x \in E} \mathrm{Inv}(\gamma_x)$$

Proposition 25 *Assume that $\mathcal{C}$ is a connectivity on E, then the following conditions are satisfied:*
(O1) every γ_x is an opening
(O2) $\gamma_x(\{x\}) = \{x\}$
(O3) $\gamma_x(X) \cap \gamma_y(X) = \varnothing$ or $\gamma_x(X) = \gamma_y(X)$
(O4) $x \notin X \Rightarrow \gamma_x(X) = \varnothing$
Conversely if γ_x, $x \in E$, is a family of operators satisfying (O1)-(O4) then $\mathcal{C} = \bigcup_{x \in E} \mathrm{Inv}(\gamma_x)$ defines a connectivity.

The principal interest of connection openings lies in the following corollary of [10]

Corollary 26 *Openings γ_x partition any $X \subseteq E$ into the smallest possible number of components belonging to the class $\mathcal{C}$.*

Given a set $X \subseteq E$, every connected component $\gamma_x(X)$ of X is called a *grain* of X. The next result ([3]) says that every connected subset of X is contained within some grain of X

Proposition 27 *Given a connectivity on E and a set $X \subseteq E$. If $C \subseteq X$ is a connected set, then C is contained within some grain of X.*

Another useful property ([10]), shows that x plays no particular role in $\gamma_x(X)$.

Corollary 28 *For all $x, y \in E$ and all $X \subseteq E$ we have*
$y \in \gamma_x(X) \Leftrightarrow \gamma_x(X) = \gamma_x(X)$ and in particular $y \in \gamma_x(X) \Leftrightarrow x \in \gamma_y(X)$

And finally the link between connective classes and partitions.

Definition 29 *Given a space E, a function $P : E \to P(E)$ is called a partition of E if*
(i) $x \in P(x)$, $x \in E$
(ii) $P(x) = P(y)$ or $P(x) \cap P(y) = \varnothing$, for $x, y \in E$

If E is endowed with a connectivity $\mathcal{C}$ and if $P(x) \in \mathcal{C}$ for every $x \in E$, then we say that the partition P is connected.

Given a connective class, every binary image (i.e.set) $X \subseteq E$ can be associated with a connected partition $P(X)$ where the zones of $P(X)$ are the grains of X and X^c. The zone of $P(X)$ containing a point p is :

$$P(X)(p) = \begin{array}{l} \gamma_p(X) \text{ if } p \in X \\ \gamma_p(X^c) \text{ if } p \notin X \end{array}$$

Corollary 30 *For all $x, y \in E$ and all $X \subseteq E$ we have* $y \in P(X)(x) \Leftrightarrow P(X)(x) = P(X)(y)$ *and in particular* $y \in P(X)(x) \Leftrightarrow x \in P(X)(y)$

Proof. If $x \in X$, $y \in P(X)(x) = \gamma_x(X) \Rightarrow P(X)(x) = \gamma_x(X) = \gamma_y(X) = P(X)(y)$ and $x \in \gamma_y(X) = P(X)(y)$
If $x \in X^c$, the proof is similar, replacing X by X^c ■

Corollary 31 *For all $x, y \in E$ and all $X \subseteq E$ we have* $y \notin P(X)(x) \Leftrightarrow P(X)(x) \cap P(X)(y) = \varnothing$

Proof. If $x \in X$ and $y \notin X$, or vice-versa, then the implication is obvious. Consider the case where x, y both belong to X or both belong to X^c. Suppose that there exists a point $z \in P(X)(x) \cap P(X)(y)$; this would imply that $P(X)(x) = P(X)(z) = P(X)(y)$ which contradicts the hypothesis ■

Connected operators

Definition 32 *An operator ψ on $\mathcal{P}(E)$ is connected if the partition $P(\psi(X))$ is coarser than $P(X)$ for every set $X \subset E$*

12.2 Connections associated to a dendrogram

12.2.1 An increasing family of connections

Consider a dendrogram $\mathcal{X}$ with a stratification index st, which induces on the points of E a partial ultrametric distance χ defined as follows, for $p, q \in E$,:

- $p \notin \text{supp}(\mathcal{X})$: $\chi(p, p) = L$ and $\chi(p, q) = L$
- for $p, q \notin \text{supp}(\mathcal{X})$: If no set of $\mathcal{X}$ contains both p and q, then $\chi(p, q) = L$.
- for $p, q \in \text{supp}(\mathcal{X})$: let A be a set of $\mathcal{X}$ containing both p and q. Thus the family $(A_i)_{i \in I}$ of sets of $\mathcal{X}$ containing both p and q has a non empty intersection, and as established above is completely ordered for $\subset$. It possesse a smallest element. The distance $\chi(p, q)$ is the stratification level of the smallest set in this family.

Properties : χ is a partial ultrametric distance as:
$\forall p, q \in E : \chi(p, q) = \chi(q, p)$
$\forall p, q, r \in E : \chi(p, q) \leq \max\{\chi(p, r), \chi(r, q)\}$

The ultrametric half-distance structures the subsets of $\mathcal{P}(E)$ by associating to each set $A \in \mathcal{P}(E)$ its diameter : $\operatorname{diam}(A) = \bigvee_{p,q \in A} \chi(p, q)$.

If $A \notin \operatorname{supp}(\mathcal{X})$, then $\operatorname{diam}(A) = L$. Obviously $\operatorname{diam}(A)$ is increasing : if $A \subset B : \operatorname{diam}(A) \leq \operatorname{diam}(B)$, since B contains more pairs of points than A.

The following lemma will establish the link with the connections.

Lemma 33 *if* $A_i \in \mathcal{P}(E)$ *and* $\bigcap_{i \in I} A_i \neq \varnothing$, *then* $\operatorname{diam}(\bigcup_{i \in I} A_i) = \bigvee_{i \in I} \operatorname{diam}(A_i)$

Proof. Two cases are to be considered:
a) there exists a point $p \notin \operatorname{supp}(\mathcal{X})$ and $p \in A_k \subset \bigcup_{i \in I} A_i$, then for any $\in E$: $\chi(p, q) = L$, and $\operatorname{diam}(\bigcup_{i \in I} A_i) = \bigvee_{i \in I} \operatorname{diam}(A_i) = \operatorname{diam}(A_k) = L$
b) $\bigcup_{i \in I} A_i \subset \operatorname{supp}(\mathcal{X})$. As $\bigcap_{i \in I} A_i \neq \varnothing$, let r be an arbitrary point in $\bigcap_{i \in I} A_i$. Then $\forall p, q \in \bigcup_{i \in I} A_i : \chi(p, q) \leq \chi(p, r) \vee \chi(r, q)$. But if $p \in A_l$, then $\chi(p, r) \leq \operatorname{diam}(A_l)$, and if $q \in A_m$, then $\chi(r, q) \leq \operatorname{diam}(A_m)$. This shows that if we consider all pairs $p, q \in \bigcup_{i \in I} A_i$ then $\chi(p, q) \leq \bigvee_{i \in I} \operatorname{diam}(A_i)$.
Inversely, as for each $k \in I : A_k \subset \bigcup_{i \in I} A_i$, and as diam is an increasing operator, we have $\operatorname{diam}(A_k) \leq \operatorname{diam}(\bigcup_{i \in I} A_i)$. Hence $\bigvee_{i \in I} \operatorname{diam}(A_i) \leq \operatorname{diam}(\bigcup_{i \in I} A_i)$.
This shows that indeed $\operatorname{diam}(\bigcup_{i \in I} A_i) = \bigvee_{i \in I} \operatorname{diam}(A_i)$. ∎

Corollary 34 *The family* $\mathcal{T}_\lambda \subseteq \mathcal{P}(E)$ *of all sets* A *verifying* $\operatorname{diam}(A_i) < \lambda \leq L$ *forms a partial connection. If the dendrogram is a (covering) hierarchy, then we get a connection.*

Proof. $A_i \in \mathcal{T}_\lambda$ and $\bigcap_{i \in I} A_i \neq \varnothing$, then $\operatorname{diam}(\bigcup_{i \in I} A_i) = \bigvee_{i \in I} \operatorname{diam}(A_i) < \lambda$, hence $\bigcup_{i \in I} A_i \in \mathcal{T}_l$ ∎

As λ increases, the family $\mathcal{T}_\lambda$ gets larger and its support also increases. The whole $\mathcal{P}(E)$ gets structured in connections with increasing indices.

In the binary case, we have a binary ultrametric half-distance. And a set verifying $\operatorname{diam}(A) < 1$ is included in one of the classes of the partition. If two sets A and B have a non empty intersection and verify $\operatorname{diam}(A) = \operatorname{diam}(B) = 0$, then they belong to the same tile of the partition as $\operatorname{diam}(A \cup B) = \operatorname{diam}(A) \vee \operatorname{diam}(B) = 0$.

12.2.2 The connected openings of origin x.

The connected opening of origin x associated to the connection $\mathcal{T}_\lambda$ has the following expression :

$$\gamma_x^\lambda(X) = \bigcup \{C : x \in C \subset \mathcal{T}_\lambda \text{ and } C \subseteq X\}$$

All C such that $x \in C \subset \mathcal{T}_\lambda$, have a non empty intersection, hence their union belongs also to $\mathcal{T}_\lambda$.

We recall that $\overset{\circ}{\mathrm{Ball}}(p,\rho) = \{q \in E \mid \chi(p,q) < \rho\}$.

Proposition 35 $\gamma_x^\lambda(X) = \overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap X$

Proof. We know that the diameter of a ball is equal to its radius. Hence $\mathrm{diam}(\overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap X) \leq \mathrm{diam}(\overset{\circ}{\mathrm{Ball}}(x,\lambda)) < \lambda$. Hence $\overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap X \in \mathcal{T}_\lambda$ and obviously belongs to $\gamma_x^\lambda(X) = \bigcup \{C : x \in C \subset \mathcal{T}_\lambda \text{ and } C \subseteq X\}$. On the other hand each set C verifying $x \in C \subset \mathcal{T}_\lambda$ is included in $\overset{\circ}{\mathrm{Ball}}(x,\lambda)$, hence the sets $\{C : x \in C \subset \mathcal{T}_\lambda \text{ and } C \subseteq X\}$ are included in $\overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap X$. This establishes the equality : $\gamma_x^\lambda(X) = \overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap X$ ■

As a consequence, we get an expression for $\overset{\circ}{\mathrm{Ball}}(x,\lambda) = \gamma_x^\lambda(E)$. This shows that the knowledge of γ_x^λ fully describes the hierarchy, as it helps reconstructing the balls $\overset{\circ}{\mathrm{Ball}}(x,\lambda)$ which are precisely the sets of the hierarchy. And inversely, knowing the balls $\overset{\circ}{\mathrm{Ball}}(x,\lambda)$ permits a direct construction of γ_x^λ.

Its expression clearly shows that $\gamma_x^\lambda(X) = \overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap X$ is an opening : it is increasing, anti-extensive and idempotent.

The property $\gamma_x^\lambda(X) \cap \gamma_y^\lambda(X) = \varnothing$ or $\gamma_x^\lambda(X) = \gamma_y^\lambda(X)$ is due to the fact that in $\overset{\circ}{\mathrm{Ball}}(x,\lambda)$, every point is centre. So if $y \in \overset{\circ}{\mathrm{Ball}}(x,\lambda)$, then $\overset{\circ}{\mathrm{Ball}}(x,\lambda) = \overset{\circ}{\mathrm{Ball}}(y,\lambda)$ and $\gamma_x^\lambda(X) = \gamma_y^\lambda(X)$. On the contrary, if $y \notin \overset{\circ}{\mathrm{Ball}}(x,\lambda)$, then $\overset{\circ}{\mathrm{Ball}}(x,\lambda) \cap \overset{\circ}{\mathrm{Ball}}(y,\lambda) = \varnothing$ and $\gamma_x^\lambda(X) \cap \gamma_y^\lambda(X) = \varnothing$.

As $\overset{\circ}{\mathrm{Ball}}(x,\lambda)$ is increasing with λ, so is $\gamma_x^\lambda(X)$.

12.2.3 An adjunction associated to the balls $\overset{\circ}{\mathrm{Ball}}(x,\lambda)$

The family of balls $\overset{\circ}{\mathrm{Ball}}(x,\mu)$ for increasing values of $\mu \leq \lambda$ is completely ordered for the inclusion. Thus, as recalled by Ch. Ronse in [7], the two following operators form an adjunction :

$\varepsilon_\lambda^x = \bigvee \left\{ \overset{\circ}{\mathrm{Ball}}(x,\mu) : \overset{\circ}{\mathrm{Ball}}(x,\mu) \subset X \right\}$ is an erosion and an opening. It is the largest ball $\overset{\circ}{\mathrm{Ball}}(x,\mu)$ centered in x included in X.

Its adjunct operator $\delta_\lambda^x = \bigwedge \left\{ \overset{\circ}{\mathrm{Ball}}(x,\mu) : X \subset \overset{\circ}{\mathrm{Ball}}(x,\mu) \right\}$ is a dilation and a closing. It is the smallest ball $\overset{\circ}{\mathrm{Ball}}(x,\mu)$ centered in x containing X.

These two operators are useful for interactive segmentation.

12.3 Taxonomies and connections

We recall once again the definition of a connection.

Definition 36 *Let E be an arbitrary nonempty set. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is called a connectivity class or connection if it satisfies*
(C1) $\varnothing \in \mathcal{C}$ and $\{x\} \in \mathcal{C}$ for $x \in E$
(C2) if $C_i \in \mathcal{C}$ and $\bigcap\limits_{i \in I} C_i \neq \varnothing$, then $\bigcup\limits_{i \in I} C_i \in \mathcal{C}$

This definition may be reformulated in the following way. We attribute to each subset $\mathcal{C} \subseteq \mathcal{P}(E)$ a binary label τ, verifying:
(A1) $\tau(\varnothing) = 1$ and $\tau(\{x\}) = 0$ for $x \in E$
(A2) if $A_i \in \mathcal{P}(E)$ and $\bigcap\limits_{i \in I} A_i \neq \varnothing$, then $\tau(\bigcup\limits_{i \in I} A_i) = \bigvee\limits_{i \in I} \tau(A_i)$

It is then easy to check that the family of all subsets with label 0 forms a connection.

We now extend this definition to taxonomies, where the labels take values in $[0, L]$.

Definition 37 *Let E be an arbitrary nonempty set. A family $\mathcal{T} \subseteq \mathcal{P}(E)$ is called taxonomy if each set A gets a label $\tau(A) \in [0, L]$ verifying*
(A1) $\tau(\varnothing) = L$
(A2) if $A_i \in \mathcal{P}(E)$ and $\bigcap\limits_{i \in I} A_i \neq \varnothing$, then $\tau(\bigcup\limits_{i \in I} A_i) = \bigvee\limits_{i \in I} \tau(A_i)$

The support of the taxonomy is the union of all sets A with a label smaller than $L : \tau(A) < L$. The taxonomy class $\mathcal{T}_\lambda$ of level λ are all sets A with a label smaller than $\lambda : \tau(A) < \lambda$. $\mathcal{T}_\lambda$ obviously forms a connection.

12.3.1 A dendrogram associated to the taxonomy $\mathcal{T}$

Proposition 38 *An ultrametric half distance is associated to each taxonomy $\mathcal{T}$. It is defined by:*
$\forall p, q \in E : \chi(p, q) = \bigwedge \{\tau(X) : X \in \mathcal{T}$ and $p, q \in X\}$. In the case where no set of $\mathcal{T}$ contains p and q, then $\chi(p, q) = \tau(\varnothing) = L$. In particular if no set of $\mathcal{T}$ contains p then $\chi(p, p) = \tau(\varnothing) = L$, and also for each $q \neq p : \chi(p, q) = \tau(\varnothing) = L$

Proof.
a) Obviously $\chi(p, q) = \chi(q, p)$
b) Consider three points $p, q, r \in E$. We have to verify the ultrametric inequality: $\chi(p, q) \leq \chi(p, r) \vee \chi(r, q)$
If no set of $\mathcal{T}$ contains r, then $\chi(p, r) = \chi(r, q) = L$ and the inequality is satisfied
If no set of $\mathcal{T}$ contains p and r (resp. r and q) then $\chi(p, r) = L$ (resp. $\chi(r, q) = L$) and the inequality is satisfied
Suppose that there exists a set X_1 of $\mathcal{T}$ containing p and r and a set X_2 of $\mathcal{T}$ containing r and q, then $X_1 \cup X_2$ contains p, q and r and $\chi(p, q) \leq \tau(X_1 \cup X_2) = \tau(X_1) \vee \tau(X_2)$
This relation remains true for all sets X_1 of $\mathcal{T}$ containing p and r and all sets

X_2 of $\mathcal{T}$ containing r and q, in particular those for which τ becomes minimal. Hence $\chi(p,q) \leq \chi(p,r) \vee \chi(r,q)$. ■

As we have associated to the taxonomy an ultrametric half-distance, all results presented above become applicable.

12.3.2 Adjacency relations

Reminder : Adjacency based connections An important subclass of connectivity classes is based on adjacency.

Definition 39 *A binary relation $\sim$ on $E \times E$ is called an adjacency relation if it is reflexive ($x \sim x$ for every x) and symmetrical ($x \sim y$ iff $y \sim x$).*

Given an adjacency relation on $E \times E$, we call $x_0, x_1, ..., x_n$ a *path* between $x = x_0 \sim x \sim .... \sim\ x_n = y$. Define $\mathcal{C}_\sim \subseteq \mathcal{P}(E)$ as the collection of all $C \in E$ such that any two points in C can be connected by a path that lies entirely in C.

Proposition 40 *If $\sim$ is an adjacency relation on $E \times E$, then $\mathcal{C}_\sim$ is a connectivity class.*

Proof. (C1) is obvious. If $C_i \in \mathcal{C}_\sim$ and $z \in \bigcap\limits_{i \in I} C_i$, we have to show that any two points x, y in $\bigcup\limits_{i \in I} C_i$ can be connected by a path that lies entirely in $\bigcup\limits_{i \in I} C_i$. There exists two indices in I such that $x \in C_{i_1}$ and $x \in C_{i_2}$. There exists a path linking x with z in C_{i_1} and a path linking z with y in C_{i_2}. The path between x and y is obtained by concatenating both paths. ■

12.3.3 Grey tone dissimilarity relations

We define a dissimilarity between neighboring points $\delta(p,q)$ verifying :

- reflexivity : $\forall p \in E : \delta(p,p) = 0$
- symmetry : $\forall p, q \in E : \delta(p,q) = \delta(q,p)$.

As an example we may consider a grey tone image defined on a grid and the following dissimilarity for neighboring pixels : $\delta(p,q) = |f_p - f_q|$. Based on this dissimilarity, we derive the same hierarchy, by very different means. The first method constructs the dendrogram based on the supremum of hierarchies. The second is generative, based on a taxonomy.

The lattice of hierarchies We extend the dissimilarity between two pixels into an ultrametric half distance :

- $\chi_{pq}(p,q) = \delta(p,q)$
- for any other couple of pixels $\chi_{pq}(s,t) = L$

It is easy to check that χ_{pq} is an ultrametric half-distance. The minimum in the lattice of ultrametric half-distances is an ultrametric half distance, called single linkage half distance. $\overbrace{\bigwedge_{p,q} \chi_{pq}}$ defines a hierarchy where the balls $\overset{\circ}{\text{Ball}}(x,\mu)$ are the lambda flat zones with slope lambda. $\overbrace{\bigwedge_{p,q} \chi_{pq}}(x,y)$ is the maximal dissimilarity on the path of smallest sup-section between x and y (see above).

A generative construction of a taxonomy We may also consider all pairs of neighboring pixels as a generative family for a taxonomy, governed by the rules given above :
(A1) $\tau(\varnothing) = L$
(A2) if $A_i \in \mathcal{P}(E)$ and $\bigcap_{i\in I} A_i \neq \varnothing$, then $\tau(\bigcup_{i\in I} A_i) = \bigvee_{i\in I} \tau(A_i)$.

The diameter of an element in the family is the maximal dissimilarity between two neighboring pixels. The associated half distance is $\forall p,q \in E$: $\chi(p,q) = \bigwedge \{\tau(X) : X \in \mathcal{T} \text{ and } p,q \in X\}$

13 Conclusion

Two trees govern multiscale mathematical morphology. On one hand the min-tree/max-tree structures the successive thresholds of an image ; on the other hand hierarchical segmentation which produces series of nested partitions.

Both are dendrograms, characterized by a simple but constraining axiom : for any set belonging to the dendrogram, $\text{Pred}(A)$ is well ordered for the inclusion order $\subset$. Partitions and partial partitions are even simpler $\text{Pred}(A) = A$. We have developed all usual concepts and tools from this simple axiom, in particular the so useful partial ultrametric distance governing the points of the domain E.

We have shown that dendrograms have the structure of a complete lattice.

The successive thresholds of a dendrogram have increasing supports. By adding the union axiom, one obtains hierarchies where the support of all thresholds is identical. If furthermore the support of a hierarchy covers the domain E, we say that it is a covering hierarchy. The catchment basins of a topographic surface, as the relief is progressively flooded, form a covering hierarchy.

The flexibility of the supports of a dendrogram permits a simple definition of erosions and dilations, the support of the resulting dendrogram increasing or decreasing as needed by the transform. Two adjunctions have been defined, a finer and a coarser one, from which openings, closings and morphological filters may be derived.

It remains now to implement these operators in order to derive the classical morphological filters based on openings and closings.

We have reinterpreted some classical tools for interactive image segmentation in the light of the structures studied above.

Finally we have defined taxonomies, extendeding to hierarchies the connections previously only defined for partitions.

References

[1] J. Angulo and D. Jeulin. Stochastic watershed segmentation. *ISMM07 : Mathematical Morphology and its applications to Signal and Image Processing*, pages 265–276, 2007.

[2] J. P. Benzécri. *L'analyse des données 1. La taxinomie*, chapter 3, pages 119–153. Dunod, 1973.

[3] H. J. A. M. Heijmans. Connected morphological operators for binary images. *Computer Vision and Image Understanding*, 73(1):99 – 120, 1999.

[4] F. Meyer. An overview of morphological segmentation. *International Journal of Pattern Recognition and Artificial Intelligence*, 17(7):1089–1118, 2001.

[5] C. Ronse. Partial partitions, partial connections and connective segmentation. *J. Math. Imaging Vis.*, 32(2):97–125, 2008.

[6] C. Ronse. Reconstructing masks from markers in non-distributive lattices. *Appl. Algebra Eng., Commun. Comput.*, 19:51–85, February 2008.

[7] C. Ronse. Adjunctions on the lattices of partitions and of partial partitions. *Appl. Algebra Eng. Commun. Comput.*, 21(5):343–396, 2010.

[8] P. Salembier and L. Garrido. Connected operators based on region-tree pruning. In J. Goutsias, L. Vincent, and D. S. Bloomberg, editors, *Mathematical Morphology and its Applications to Image and Signal Processing*, volume 18, pages 169–178. Springer US, 2002.

[9] L. Schwartz. *Analyse : topologie générale et analyse fonctionnelle*, chapter 3, pages 233–240. Hermann, 2008.

[10] J. Serra, editor. *Image Analysis and Mathematical Morphology. II: Theoretical Advances*. Academic Press, London, 1988.

[11] J. Serra. Morphological operators for the segmentation of colour images. In M. S. M. Bilodeau, F. Meyer, editor, *Space, structure, and randomness. Contributions in honor of Georges Matheron in the fields of geostatistics, random sets, and mathematical morphology.*, pages 223–255. Lecture Notes in Statistics 183. New York, NY: Springer. xviii, 395 p., 2005.

[12] J. Serra. A lattice approach to image segmentation. *J. Math. Imaging Vis.*, 24:83–130, January 2006.

[13] M. F. Zanoguera, B. Marcotegui, and F. Meyer. An interactive colour image segmentation system. In *Wiamis'99 : Workshop on Image Analysis for Multimedia Interactive Services*, pages 137–141. Heinrich-Hertz Institut Berlin, 1999.